

Linear and Quadratic Inequalities

The solution to a problem may be not a single value, but a range of values. A chemical engineer may need a reaction to occur within a certain time frame in order to reduce undesired pollutants. An architect may design a building to deflect less than a given distance in a strong wind. A doctor may choose a dose of medication so that a safe but effective level remains in the body after a specified time.

These situations illustrate the importance of inequalities. While there may be many acceptable values in each of the scenarios above, in each case there is a lower acceptable limit, an upper acceptable limit, or both. Even though many solutions exist, we still need accurate mathematical models and methods to obtain the solutions.

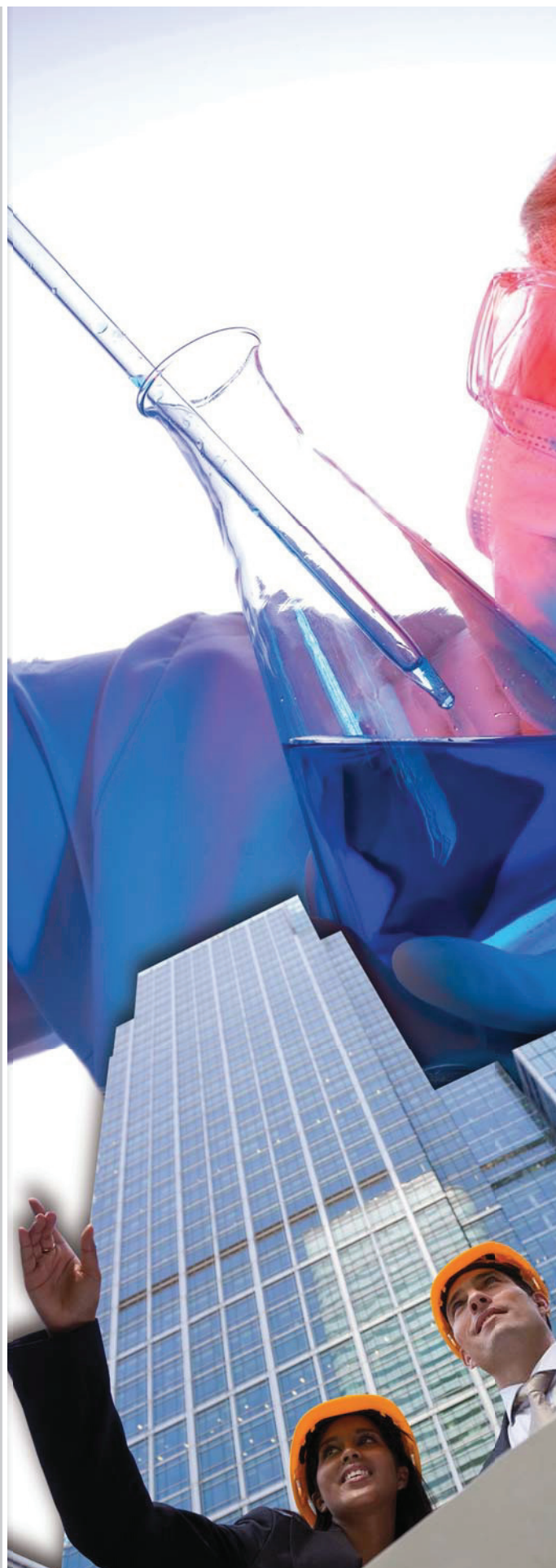
Web Link

A small number of mathematicians have earned the distinction of having an inequality named for them. To learn more about these special inequalities, go to www.mhrprecalc11.ca and follow the links.

Key Terms

solution region
boundary

test point



9.1

Linear Inequalities in Two Variables

Focus on...

- explaining when a solid or a dashed line should be used in the solution to an inequality
- explaining how to use test points to find the solution to an inequality
- sketching, with or without technology, the graph of a linear inequality
- solving a problem that involves a linear inequality



How can you choose the correct amounts of two items when both items are desirable? Suppose you want to take music lessons, but you also want to work out at a local gym. Your budget limits the amount you can spend. Solving a linear inequality can show you the alternatives that will help you meet both your musical and fitness goals and stay within your budget. Linear inequalities can model this situation and many others that require you to choose from combinations of two or more quantities.

Investigate Linear Inequalities

Materials

- grid paper
- straight edge

Suppose that you have received a gift card for a music-downloading service. The card has a value of \$15. You have explored the Web site and discovered that individual songs cost \$1 each and a complete album costs \$5. Both prices include all taxes. Work with a partner to investigate this situation.

1. List all possible combinations of songs and albums that you can purchase if you spend all \$15 of your gift card.
2. Let x represent the number of individual songs purchased and y represent the number of albums purchased. Write a linear equation in two variables to model the situation described in step 1.
3. Plot the points from step 1 that represent the coordinates of a combination of songs and albums that you can purchase for \$15. On the same coordinate grid, graph the linear equation from step 2.

4. List all possible combinations of songs and albums that you can purchase for less than or equal to the total amount of your gift card.
5. Write a linear inequality in two variables to model the situation described in step 4.
6. Verify the combinations you found in step 4 by substituting the values in the inequality you wrote in step 5.
7. Compare your work with that of another pair of students to see if you agree on the possible combinations and the inequality that models the situation.
8. On the coordinate grid from step 3, plot each point that represents the coordinates of a combination of songs and albums that you can purchase for less than or equal to \$15.

Is it convenient to find all possible combinations this way?

How is the value of the gift card reflected in your inequality?

Reflect and Respond

9. How does the graph show that it is possible to spend the entire value of the gift card?
10. Consider the inequality you wrote in step 5. Is it represented on your graph? Explain.
11. How would your graph change if the variables x and y represented quantities that could be real numbers, rather than whole numbers?

How are the real numbers different from the whole numbers?

Web Link

Linear programming is a mathematical method of finding the best solution to a problem requiring a combination of two different items. Linear programming is part of the mathematical field of operations research. To learn more about a career as an operations researcher, go to www.mhrprecalc11.ca and follow the links.

Link the Ideas

A linear inequality in two variables may be in one of the following four forms:

- $Ax + By < C$
- $Ax + By \leq C$
- $Ax + By > C$
- $Ax + By \geq C$

where A , B , and C are real numbers.

An inequality in the two variables x and y describes a region in the Cartesian plane. The ordered pair (x, y) is a solution to a linear inequality if the inequality is true when the values of x and y are substituted into the inequality. The set of points that satisfy a linear inequality can be called the solution set, or **solution region**.

solution region

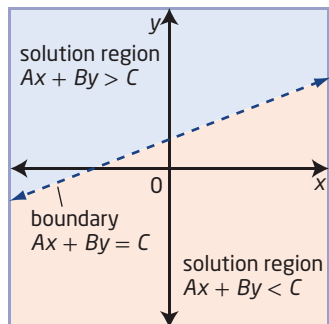
- all the points in the Cartesian plane that satisfy an inequality
- also known as the solution set

boundary

- a line or curve that separates the Cartesian plane into two regions
- may or may not be part of the solution region
- drawn as a solid line and included in the solution region if the inequality involves \leq or \geq
- drawn as a dashed line and not included in the solution region if the inequality involves $<$ or $>$

The line related to the linear equality $Ax + By = C$, or **boundary**, divides the Cartesian plane into two solution regions.

- For one solution region, $Ax + By > C$ is true.
- For the other solution region, $Ax + By < C$ is true.



In your previous study of linear equations in two variables, the solution was all the ordered pairs located on the graph of the line. The solution to a linear inequality in two variables is a solution region that may or may not include the line, depending on the inequality.

Example 1

Graph a Linear Inequality of the Form $Ax + By \leq C$

- Graph $2x + 3y \leq 6$.
- Determine if the point $(-2, 4)$ is part of the solution.

Solution

- First, determine the boundary of the graph, and then determine which region contains the solution.

There are several approaches to graphing the boundary.

Method 1: Solve for y

Solve the inequality for y in terms of x .

$$2x + 3y \leq 6$$

$$3y \leq -2x + 6$$

$$y \leq -\frac{2}{3}x + 2$$

Since the inequality symbol is \leq , points on the boundary are included in the solution. Use the slope of $-\frac{2}{3}$ and the y -intercept of 2 to graph the related line $y = -\frac{2}{3}x + 2$ as a solid line.

Method 2: Use the Intercepts

Since the inequality symbol is \leq , points on the boundary are included in the solution.

Use the intercepts to graph the related line $2x + 3y = 6$ as a solid line.

For $x = 0$:

$$2(0) + 3y = 6$$

$$3y = 6$$

$$y = 2$$

For $y = 0$:

$$2x + 3(0) = 6$$

$$2x = 6$$

$$x = 3$$

Locate the points $(0, 2)$ and $(3, 0)$ and draw a line passing through them.

After graphing the boundary, select a **test point** from each region to determine which contains the solution.

Why must the test point not be on the line?

test point

- a point not on the boundary of the graph of an inequality that is representative of all the points in a region
- a point that is used to determine whether the points in a region satisfy the inequality

For $(0, 0)$:

Left Side

$$2x + 3y$$

$$= 2(0) + 3(0)$$

$$= 0$$

Right Side

$$6$$

$$\text{Left Side} \leq \text{Right Side}$$

For $(2, 4)$:

Left Side

$$2x + 3y$$

$$= 2(2) + 3(4)$$

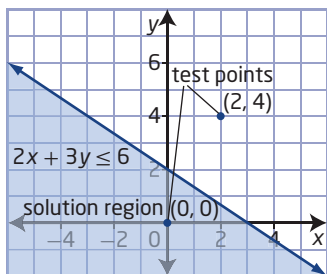
$$= 16$$

Right Side

$$6$$

$$\text{Left Side} \not\leq \text{Right Side}$$

The point $(0, 0)$ satisfies the inequality, so shade that region as the solution region.



b) Determine if the point $(-2, 4)$ is in the solution region.

Left Side

$$2x + 3y$$

$$= 2(-2) + 3(4)$$

$$= -4 + 12$$

$$= 8$$

Right Side

$$6$$

$$\text{Left Side} \not\leq \text{Right Side}$$

The point $(-2, 4)$ is not part of the solution to the inequality $2x + 3y \leq 6$. From the graph of $2x + 3y \leq 6$, the point $(-2, 4)$ is not in the solution region.

Your Turn

a) Graph $4x + 2y \geq 10$.

b) Determine if the point $(1, 3)$ is part of the solution.

Example 2

Graph a Linear Inequality of the Form $Ax + By > C$

Graph $10x - 5y > 0$.

Solution

Solve the inequality for y in terms of x .

Is there another way to solve the inequality?

$$\begin{aligned}10x - 5y &> 0 \\ -5y &> -10x \\ y &< 2x\end{aligned}$$

Why is the inequality symbol reversed?

Graph the related line $y = 2x$ as a broken, or dashed, line.

Use a test point from one region. Try $(-2, 3)$.

Left Side

Right Side

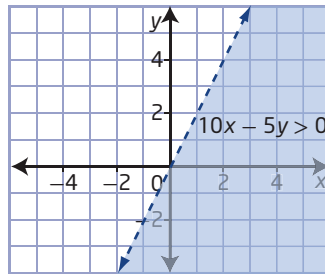
$$\begin{aligned}10x - 5y \\ = 10(-2) - 5(3) \\ = -20 - 15 \\ = -35\end{aligned}$$

$$0$$

Why is the point $(0, 0)$ not used as a test point this time?

Left Side \nless Right Side

The point $(-2, 3)$ does not satisfy the inequality. Shade the other region as the solution region.



Why is the boundary graphed with a dashed line?

Did You Know?

An open solution region does not include any of the points on the line that bounds it. A closed solution region includes the points on the line that bounds it.

Verify the solution region by using a test point in the shaded region.

Try $(2, -3)$.

Left Side

Right Side

$$\begin{aligned}10x - 5y \\ = 10(2) - 5(-3) \\ = 20 + 15 \\ = 35\end{aligned}$$

$$0$$

Left Side $>$ Right Side

The graph of the solution region is correct.

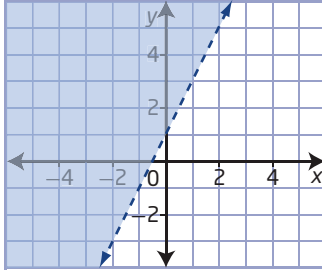
Your Turn

Graph $5x - 20y < 0$.

Example 3

Write an Inequality Given Its Graph

Write an inequality to represent the graph.



Solution

Write the equation of the boundary in slope-intercept form, $y = mx + b$.

The y -intercept is 1. So, $b = 1$.

Use the points $(0, 1)$ and $(1, 3)$ to determine that the slope, m , is 2.

$$y = 2x + 1$$

The boundary is a dashed line, so it is not part of the solution region.

Use a test point from the solution region to determine whether the inequality symbol is $>$ or $<$.

Try $(-2, 4)$.

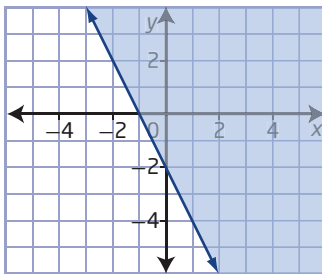
Left Side	Right Side
y	$2x + 1$
$= 4$	$= 2(-2) + 1$
	$= -3$

$$\text{Left Side} > \text{Right Side}$$

An inequality that represents the graph is $y > 2x + 1$.

Your Turn

Write an inequality to represent the graph.



Example 4

Write and Solve an Inequality

Suppose that you are constructing a tabletop using aluminum and glass. The most that you can spend on materials is \$50. Laminated safety glass costs \$60/m², and aluminum costs \$1.75/ft. You can choose the dimensions of the table and the amount of each material used. Find all possible combinations of materials sufficient to make the tabletop.



Solution

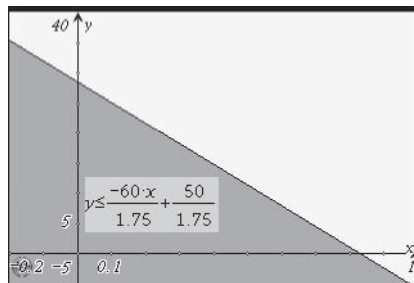
Let x represent the area of glass used and y represent the length of aluminum used. Then, the inequality representing this situation is

$$60x + 1.75y \leq 50$$

Solve the inequality for y in terms of x .

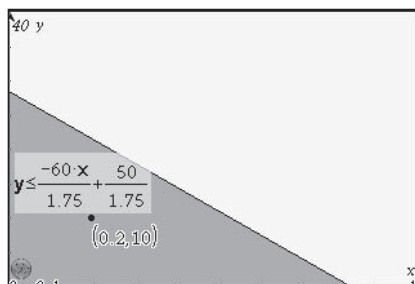
$$\begin{aligned} 60x + 1.75y &\leq 50 \\ 1.75y &\leq -60x + 50 \\ y &\leq \frac{-60x}{1.75} + \frac{50}{1.75} \end{aligned}$$

Use graphing technology to graph the related line $y = -\frac{60}{1.75}x + \frac{50}{1.75}$ as a solid line. Shade the region where a test point results in a true statement as the solution region.



Examine the solution region.

You cannot have a negative amount of safety glass or aluminum. Therefore, the domain and range contain only non-negative values.



The graph shows all possible combinations of glass and aluminum that can be used for the tabletop. One possible solution is $(0.2, 10)$. This represents 0.2 m^2 of the laminated safety glass and 10 ft of aluminum.

Your Turn

Use technology to find all possible combinations of tile and stone that can be used to make a mosaic. Tile costs $\$2.50/\text{ft}^2$, stone costs $\$6/\text{kg}$, and the budget for the mosaic is $\$150$.

Key Ideas

- A linear inequality in two variables describes a region of the Cartesian plane.
- All the points in one solution region satisfy the inequality and make up the solution region.
- The boundary of the solution region is the graph of the related linear equation.
 - When the inequality symbol is \leq or \geq , the points on the boundary are included in the solution region and the line is a solid line.
 - When the inequality symbol is $<$ or $>$, the points on the boundary are not included in the solution region and the line is a dashed line.
- Use a test point to determine which region is the solution region for the inequality.

Check Your Understanding

Practise

1. Which of the ordered pairs are solutions to the given inequality?

- a) $y < x + 3$,
 $\{(7, 10), (-7, 10), (6, 7), (12, 9)\}$
- b) $-x + y \leq -5$,
 $\{(2, 3), (-6, -12), (4, -1), (8, -2)\}$
- c) $3x - 2y > 12$,
 $\{(6, 3), (12, -4), (-6, -3), (5, 1)\}$
- d) $2x + y \geq 6$,
 $\{(0, 0), (3, 1), (-4, -2), (6, -4)\}$

2. Which of the ordered pairs are *not* solutions to the given inequality?

- a) $y > -x + 1$,
 $\{(1, 0), (-2, 1), (4, 7), (10, 8)\}$
- b) $x + y \geq 6$,
 $\{(2, 4), (-5, 8), (4, 1), (8, 2)\}$
- c) $4x - 3y < 10$,
 $\{(1, 3), (5, 1), (-2, -3), (5, 6)\}$
- d) $5x + 2y \leq 9$,
 $\{(0, 0), (3, -1), (-4, 2), (1, -2)\}$

3. Consider each inequality.

- Express y in terms of x , if necessary. Identify the slope and the y -intercept.
- Indicate whether the boundary should be a solid line or a dashed line.

- a) $y \leq x + 3$
- b) $y > 3x + 5$
- c) $4x + y > 7$
- d) $2x - y \leq 10$
- e) $4x + 5y \geq 20$
- f) $x - 2y < 10$

4. Graph each inequality without using technology.

- a) $y \leq -2x + 5$
- b) $3y - x > 8$
- c) $4x + 2y - 12 \geq 0$
- d) $4x - 10y < 40$
- e) $x \geq y - 6$

5. Graph each inequality using technology.

- a) $6x - 5y \leq 18$
- b) $x + 4y < 30$
- c) $-5x + 12y - 28 > 0$
- d) $x \leq 6y + 11$
- e) $3.6x - 5.3y + 30 \geq 4$

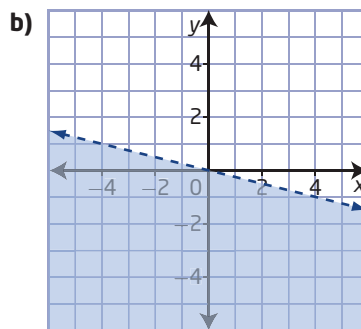
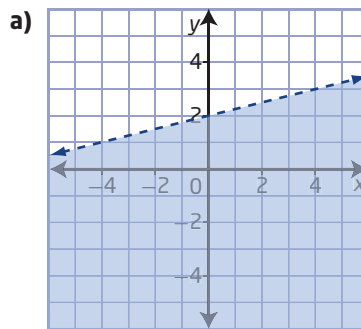
6. Determine the solution to $-5y \leq x$.

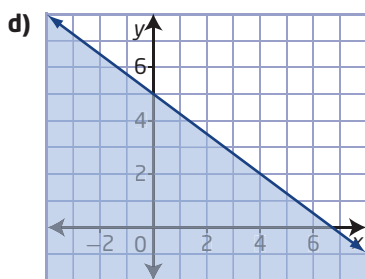
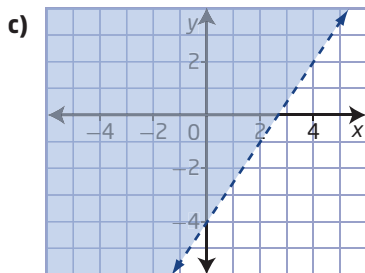
7. Use graphing technology to determine the solution to $7x - 2y > 0$.

8. Graph each inequality. Explain your choice of graphing methods.

- a) $6x + 3y \geq 21$
- b) $10x < 2.5y$
- c) $2.5x < 10y$
- d) $4.89x + 12.79y \leq 145$
- e) $0.8x - 0.4y > 0$

9. Determine the inequality that corresponds to each graph.





Apply

10. Express the solution to $x + 0y > 0$ graphically and in words.
11. Amaruq has a part-time job that pays her \$12/h. She also sews baby moccasins and sells them for a profit of \$12 each. Amaruq wants to earn at least \$250/week.



- Write an inequality that represents the number of hours that Amaruq can work and the number of baby moccasins she can sell to earn at least \$250. Include any restrictions on the variables.
- Graph the inequality.
- List three different ordered pairs in the solution.
- Give at least one reason that Amaruq would want to earn income from her part-time job as well as her sewing business, instead of focusing on one method only.

12. The Alberta Foundation for the Arts provides grants to support artists. The Aboriginal Arts Project Grant is one of its programs. Suppose that Camille has received a grant and is to spend at most \$3000 of the grant on marketing and training combined. It costs \$30/h to work with an elder in a mentorship program and \$50/h for marketing assistance.
- Write an inequality to represent the number of hours working with an elder and receiving marketing assistance that Camille can afford. Include any restrictions on the variables.
 - Graph the inequality.



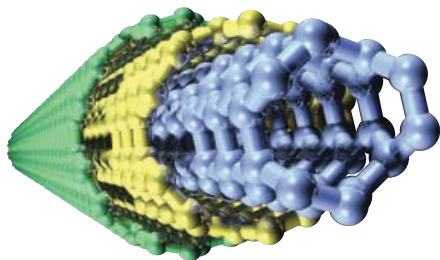
Mother Eagle by Jason Carter, artist chosen to represent Alberta at the Vancouver 2010 Olympics. Jason is a member of the Little Red River Cree Nation.

Web Link

To learn more about the Alberta Foundation for the Arts, go to www.mhrprecalc11.ca and follow the links.

13. Mariya has purchased a new smart phone and is trying to decide on a service plan. Without a plan, each minute of use costs \$0.30 and each megabyte of data costs \$0.05. A plan that allows unlimited talk and data costs \$100/month. Under which circumstances is the plan a better choice for Mariya?

14. Suppose a designer is modifying the tabletop from Example 4. The designer wants to replace the aluminum used in the table with a nanomaterial made from nanotubes. The budget for the project remains \$50, the cost of glass is still \$60/m², and the nanomaterial costs \$45/kg. Determine all possible combinations of material available to the designer.



Multi-walled carbon nanotube

15. Speed skaters spend many hours training on and off the ice to improve their strength and conditioning. Suppose a team has a monthly training budget of \$7000. Ice rental costs \$125/h, and gym rental for strength training costs \$55/h. Determine the solution region, or all possible combinations of training time that the team can afford.



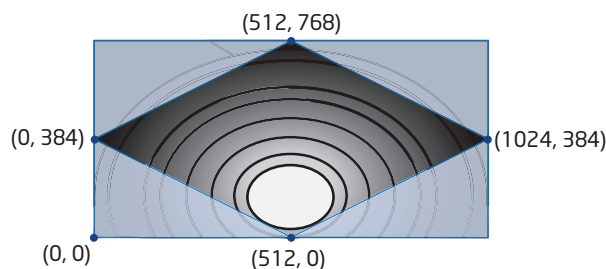
Olympic gold medalist
Christine Nesbitt

Did You Know?

Canadian long-track and short-track speed skaters won 10 medals at the 2010 Olympic Winter Games in Vancouver, part of an Olympic record for the most gold medals won by a country in the history of the Winter Games.

Extend

16. Drawing a straight line is not the only way to divide a plane into two regions.
- Determine one other relation that when graphed divides the Cartesian plane into two regions.
 - For your graph, write inequalities that describe each region of the Cartesian plane in terms of your relation. Justify your answer.
 - Does your relation satisfy the definition of a solution region? Explain.
17. Masha is a video game designer. She treats the computer screen like a grid. Each pixel on the screen is represented by a coordinate pair, with the pixel in the bottom left corner of the screen as (0, 0). For one scene in a game she is working on, she needs to have a background like the one shown.



The shaded region on the screen is made up of four inequalities. What are the four inequalities?

18. **MINI LAB** Work in small groups.

In April 2008, Manitoba Hydro agreed to provide Wisconsin Public Service with up to 500 MW (megawatts) of hydroelectric power over 15 years, starting in 2018. Hydroelectric projects generate the majority of power in Manitoba; however, wind power is a method of electricity generation that may become more common. Suppose that hydroelectric power costs \$60/MWh (megawatt hour) to produce, wind power costs \$90/MWh, and the total budget for all power generation is \$35 000/h.

Step 1 Write the inequality that represents the cost of power generation. Let x represent the number of megawatt hours of hydroelectric power produced. Let y represent the number of megawatt hours of wind power produced.

Step 2 Graph and solve the inequality for the cost of power generation given the restrictions imposed by the hydroelectric agreement. Determine the coordinates of the vertices of the solution region. Interpret the intercepts in the context of this situation.

Step 3 Suppose that Manitoba Hydro can sell the hydroelectric power for \$95/MWh and the wind power for \$105/MWh. The equation $R = 95x + 105y$ gives the revenue, R , in dollars, from the sale of power. Use a spreadsheet to find the revenue for a number of different points in the solution region. Is it possible to find the revenue for all possible combinations of power generation? Can you guarantee that the point giving the maximum possible revenue is shown on your spreadsheet?

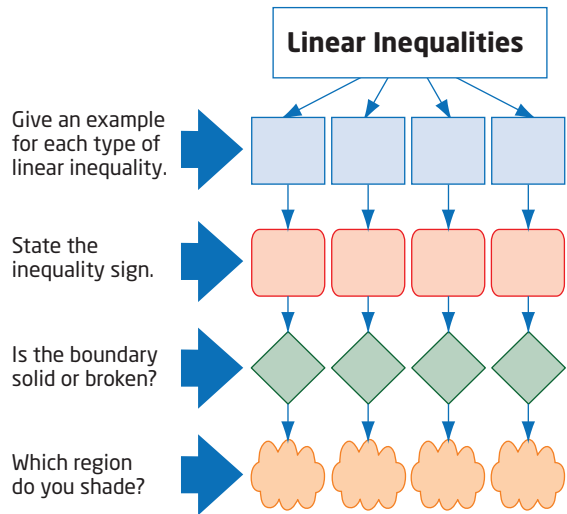
	A	B	C
1	Hydroelectric (MWH)	Wind (MWH)	Revenue (\$)
2	0	0	0
3	50	25	7375
4	100	50	14750
5	150	75	22125
6	200	100	29500
7			
8			

Step 4 It can be shown that the maximum revenue is always obtained from one of the vertices of the solution region. What combination of wind and hydroelectric power leads to the highest revenue?

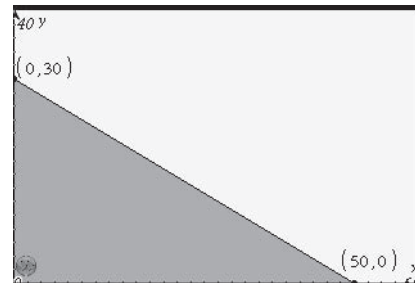
Step 5 With your group, discuss reasons that a combination other than the one that produces the maximum revenue might be chosen.

Create Connections

19. Copy and complete the following mind map.



20. The graph shows the solution to a linear inequality.



a) Write a scenario that has this region as its solution. Justify your answer.

b) Exchange your scenario with a partner. Verify that the given solution fits each scenario.

21. The inequality $2x - 3y + 24 > 0$, the positive y -axis, and the negative x -axis define a region in quadrant II.

a) Determine the area of this region.

b) How does the area of this region depend on the y -intercept of the boundary of the inequality $2x - 3y + 24 > 0$?

c) How does the area of this region depend on the slope of the boundary of the inequality $2x - 3y + 24 > 0$?

d) How would your answers to parts b) and c) change for regions with the same shape located in the other quadrants?

Quadratic Inequalities in One Variable

Focus on...

- developing strategies to solve quadratic inequalities in one variable
- modelling and solving problems using quadratic inequalities
- interpreting quadratic inequalities to determine solutions to problems

An engineer designing a roller coaster must know the minimum speed required for the cars to stay on the track. To determine this value, the engineer can solve a quadratic inequality. While infinitely many answers are possible, it is important that the engineer be sure that the speed of the car is in the solution region.

A bicycle manufacturer must know the maximum distance the rear suspension will travel when going over rough terrain. For many bicycles, the movement of the rear wheel is described by a quadratic equation, so this problem requires the solution to a quadratic inequality. Solving quadratic inequalities is important to ensure that the manufacturer can reduce warranty claims.

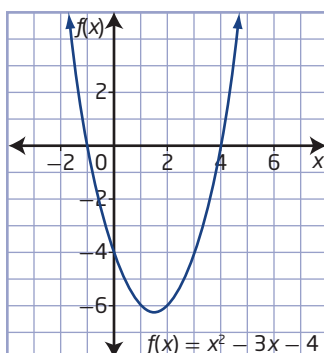


Investigate Quadratic Inequalities

Materials

- grid paper
- coloured pens, pencils, or markers

1. Consider the quadratic inequalities $x^2 - 3x - 4 > 0$ and $x^2 - 3x - 4 < 0$.
 - a) Use the graph of the corresponding function $f(x) = x^2 - 3x - 4$ to identify the zeros of the function.



The x-axis is divided into three sections by the parabola. What are the three sections?

- b) Identify the x -values for which the inequality $x^2 - 3x - 4 > 0$ is true.
- c) Identify the x -values for which the inequality $x^2 - 3x - 4 < 0$ is true.
2. Consider the quadratic inequality $x^2 - x - 6 < 0$.
- a) Graph the corresponding quadratic function $f(x) = x^2 - x - 6$.
- b) How many zeros does the function have?
- c) Colour the portion of the x -axis for which the inequality $x^2 - x - 6 < 0$ is true.
- d) Write one or more inequalities to represent the values of x for which the function is negative. Show these values on a number line.
3. Consider the quadratic inequality $x^2 - 4x + 4 > 0$.
- a) Graph the corresponding quadratic function $f(x) = x^2 - 4x + 4$.
- b) How many zeros does the function have?
- c) Colour the portion of the x -axis for which the inequality $x^2 - 4x + 4 > 0$ is true.
- d) Write one or more inequalities to represent the values of x for which the function is positive. Show these values on a number line.

Reflect and Respond

4. a) Explain how you arrived at the inequalities in steps 2d) and 3d).
- b) What would you look for in the graph of the related function when solving a quadratic inequality of the form $ax^2 + bx + c > 0$ or $ax^2 + bx + c < 0$?

Did You Know?

Babylonian mathematicians were among the first to solve quadratics. However, they had no notation for variables, equations, or inequalities, and did not understand negative numbers. It was more than 1500 years before notation was developed.

Link the Ideas

You can write quadratic inequalities in one variable in one of the following four forms:

- $ax^2 + bx + c < 0$
- $ax^2 + bx + c \leq 0$
- $ax^2 + bx + c > 0$
- $ax^2 + bx + c \geq 0$

where a , b , and c are real numbers and $a \neq 0$.

You can solve quadratic inequalities graphically or algebraically. The solution set to a quadratic inequality in one variable can have no values, one value, or an infinite number of values.

Example 1

Solve a Quadratic Inequality of the Form $ax^2 + bx + c \leq 0$, $a > 0$

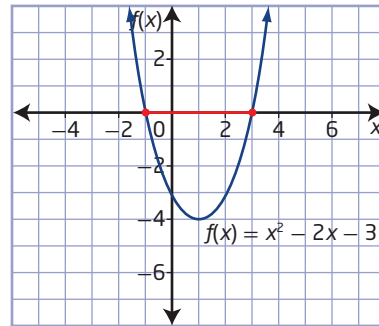
Solve $x^2 - 2x - 3 \leq 0$.

Solution

Method 1: Graph the Corresponding Function

Graph the corresponding function $f(x) = x^2 - 2x - 3$.

To determine the solution to $x^2 - 2x - 3 \leq 0$, look for the values of x for which the graph of $f(x)$ lies on or below the x -axis.



What strategies can you use to sketch the graph of a quadratic function in standard form?

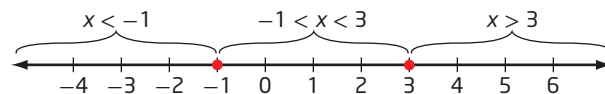
The parabola lies on the x -axis at $x = -1$ and $x = 3$. The graph lies below the x -axis between these values of x . Therefore, the solution set is all real values of x between -1 and 3 , inclusive, or $\{x \mid -1 \leq x \leq 3, x \in \mathbb{R}\}$.

Method 2: Roots and Test Points

Solve the related equation $x^2 - 2x - 3 = 0$ to find the roots. Then, use a number line and test points to determine the intervals that satisfy the inequality.

$$\begin{aligned}x^2 - 2x - 3 &= 0 \\(x + 1)(x - 3) &= 0 \\x + 1 = 0 \quad \text{or} \quad x - 3 &= 0 \\x = -1 \quad \quad \quad x &= 3\end{aligned}$$

Plot -1 and 3 on a number line. Use closed circles since these values are solutions to the inequality.



Does it matter which values you choose as test points?

Are there any values that you should not choose?

The x -axis is divided into three intervals by the roots of the equation. Choose one test point from each interval, say -2 , 0 , and 5 . Then, substitute each value into the quadratic inequality to determine whether the result satisfies the inequality.

Use a table to organize the results.

Interval	$x < -1$	$-1 < x < 3$	$x > 3$
Test Point	-2	0	5
Substitution	$(-2)^2 - 2(-2) - 3$ $= 4 + 4 - 3$ $= 5$	$0^2 - 2(0) - 3$ $= 0 + 0 - 3$ $= -3$	$5^2 - 2(5) - 3$ $= 25 - 10 - 3$ $= 12$
Is $x^2 - 2x - 3 \leq 0$?	no	yes	no

The values of x between -1 and 3 also satisfy the inequality.

The value of $x^2 - 2x - 3$ is negative in the interval $-1 < x < 3$.

The solution set is $\{x \mid -1 \leq x \leq 3, x \in \mathbb{R}\}$.



Method 3: Case Analysis

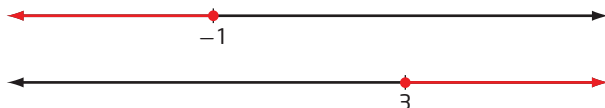
Factor the quadratic expression to rewrite the inequality as $(x + 1)(x - 3) \leq 0$.

The product of two factors is negative when the factors have different signs. There are two ways for this to happen.

Case 1: The first factor is negative and the second factor is positive.

$$x + 1 \leq 0 \text{ and } x - 3 \geq 0$$

Solve these inequalities to obtain $x \leq -1$ and $x \geq 3$.



Any x -values that satisfy both conditions are part of the solution set. There are no values that make both of these inequalities true.

Why are there no values that make both inequalities true?

Case 2: The first factor is positive and the second factor is negative.

$$x + 1 \geq 0 \text{ and } x - 3 \leq 0$$

Solve these inequalities to obtain $x \geq -1$ and $x \leq 3$.



The dashed lines indicate that $-1 \leq x \leq 3$ is common to both.

These inequalities are both true for all values between -1 and 3 , inclusive.

The solution set is $\{x \mid -1 \leq x \leq 3, x \in \mathbb{R}\}$.

How would the steps in this method change if the original inequality were $x^2 - 2x - 3 \geq 0$?

Your Turn

Solve $x^2 - 10x + 16 \leq 0$ using two different methods.

Example 2

Solve a Quadratic Inequality of the Form $ax^2 + bx + c < 0$, $a < 0$

Solve $-x^2 + x + 12 < 0$.

Solution

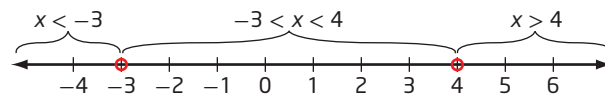
Method 1: Roots and Test Points

Solve the related equation $-x^2 + x + 12 = 0$ to find the roots.

$$\begin{aligned} -x^2 + x + 12 &= 0 \\ -1(x^2 - x - 12) &= 0 \\ -1(x + 3)(x - 4) &= 0 \\ x + 3 = 0 &\quad \text{or} \quad x - 4 = 0 \\ x = -3 &\quad \quad \quad x = 4 \end{aligned}$$

Plot -3 and 4 on a number line.

Use open circles, since these values are not solutions to the inequality.



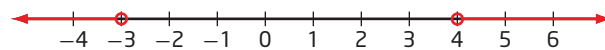
Choose a test point from each of the three intervals, say -5 , 0 , and 5 , to determine whether the result satisfies the quadratic inequality.

Use a table to organize the results.

Interval	$x < -3$	$-3 < x < 4$	$x > 4$
Test Point	-5	0	5
Substitution	$\begin{aligned} -(-5)^2 + (-5) + 12 \\ = -25 - 5 + 12 \\ = -18 \end{aligned}$	$\begin{aligned} -0^2 + 0 + 12 \\ = 0 + 0 + 12 \\ = 12 \end{aligned}$	$\begin{aligned} -5^2 + 5 + 12 \\ = -25 + 5 + 12 \\ = -8 \end{aligned}$
Is $-x^2 + x + 12 < 0$?	yes	no	yes

The values of x less than -3 or greater than 4 satisfy the inequality.

The solution set is $\{x \mid x < -3 \text{ or } x > 4, x \in \mathbb{R}\}$.



Method 2: Sign Analysis

Factor the quadratic expression to rewrite the inequality as $-1(x + 3)(x - 4) < 0$.

Determine when each of the factors, $-1(x + 3)$ and $x - 4$, is positive, zero, or negative.

Substituting -4 in $-1(x + 3)$ results in a positive value (+).

$$\begin{aligned} -1(-4 + 3) &= -1(-1) \\ &= 1 \end{aligned}$$

Substituting -3 in $-1(x + 3)$ results in a value of zero (0).

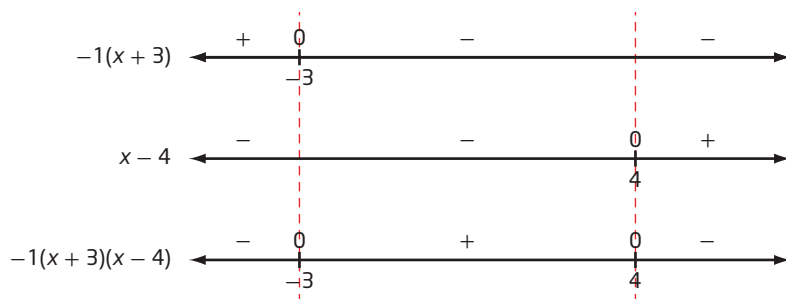
$$\begin{aligned} -1(-3 + 3) &= -1(0) \\ &= 0 \end{aligned}$$

Substituting 1 in $-1(x + 3)$ results in a negative value (-).

$$\begin{aligned} -1(1 + 3) &= -1(4) \\ &= -4 \end{aligned}$$

Since -1 is a constant factor, combine it with $(x + 3)$ to form one factor.

Sketch number lines to show the results.



From the number line representing the product, the values of x less than -3 or greater than 4 satisfy the inequality $-1(x + 3)(x - 4) < 0$.

The solution set is $\{x \mid x < -3 \text{ or } x > 4, x \in \mathbb{R}\}$.

Your Turn

Solve $-x^2 + 3x + 10 < 0$ using two different methods.

Why is it important to rewrite the inequality with 0 on one side of the inequality?

Why use the quadratic formula in this case?

Example 3

Solve a Quadratic Inequality in One Variable

Solve $2x^2 - 7x > 12$.

Solution

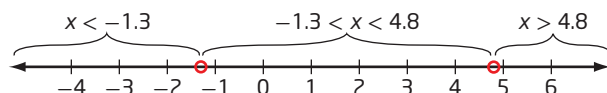
First, rewrite the inequality as $2x^2 - 7x - 12 > 0$.

Solve the related equation $2x^2 - 7x - 12 = 0$ to find the roots.

Use the quadratic formula with $a = 2$, $b = -7$, and $c = -12$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ x &= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(2)(-12)}}{2(2)} \\ x &= \frac{7 \pm \sqrt{145}}{4} \\ x &= \frac{7 + \sqrt{145}}{4} \quad \text{or} \quad x = \frac{7 - \sqrt{145}}{4} \\ x &\approx 4.8 \quad \quad \quad x \approx -1.3 \end{aligned}$$

Use a number line and test points.



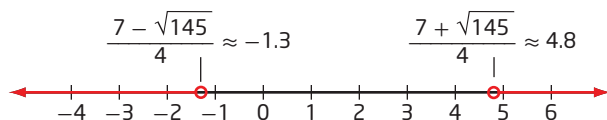
Choose a test point from each of the three intervals, say -3 , 0 , and 6 , to determine whether the results satisfy the original quadratic inequality.

Use a table to organize the results.

Interval	$x < \frac{7 - \sqrt{145}}{4}$	$\frac{7 - \sqrt{145}}{4} < x < \frac{7 + \sqrt{145}}{4}$	$x > \frac{7 + \sqrt{145}}{4}$
Test Point	-3	0	6
Substitution	$2(-3)^2 - 7(-3)$ $= 18 + 21$ $= 39$	$2(0)^2 - 7(0)$ $= 0 + 0$ $= 0$	$2(6)^2 - 7(6)$ $= 72 - 42$ $= 30$
Is $2x^2 - 7x > 12$?	yes	no	yes

Therefore, the exact solution set is

$$\left\{ x \mid x < \frac{7 - \sqrt{145}}{4} \text{ or } x > \frac{7 + \sqrt{145}}{4}, x \in \mathbb{R} \right\}.$$



Can you solve this inequality using sign analysis and case analysis?

Your Turn

Solve $x^2 - 4x > 10$.

Example 4

Apply Quadratic Inequalities

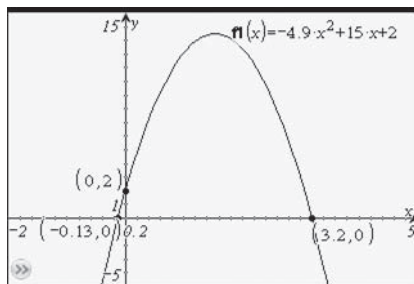
If a baseball is thrown at an initial speed of 15 m/s from a height of 2 m above the ground, the inequality $-4.9t^2 + 15t + 2 > 0$ models the time, t , in seconds, that the baseball is in flight. During what time interval is the baseball in flight?

Why is the quadratic expression greater than zero?

Solution

The baseball will be in flight from the time it is thrown until it lands on the ground.

Graph the corresponding quadratic function and determine the coordinates of the x -intercepts and the y -intercept.



Why is it useful to know the y -intercept of the graph in this case?

The graph of the function lies on or above the x -axis for values of x between approximately -0.13 and 3.2 , inclusive. However, you cannot have a negative time that the baseball will be in the air.

The solution set to the problem is $\{t \mid 0 < t < 3.2, t \in \mathbb{R}\}$. In other words, the baseball is in flight between 0 s and approximately 3.2 s after it is thrown.

Your Turn

Suppose a baseball is thrown from a height of 1.5 m. The inequality $-4.9t^2 + 17t + 1.5 > 0$ models the time, t , in seconds, that the baseball is in flight. During what time interval is the baseball in flight?

Web Link

To learn about baseball in Canada, go to www.mhrprecalc11.ca and follow the links.

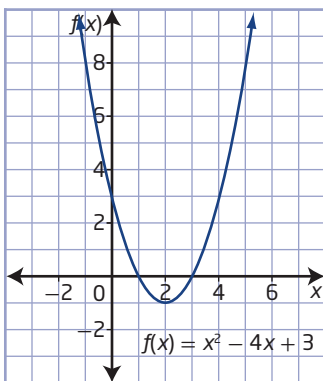
Key Ideas

- The solution to a quadratic inequality in one variable is a set of values.
- To solve a quadratic inequality, you can use one of the following strategies:
 - Graph the corresponding function, and identify the values of x for which the function lies on, above, or below the x -axis, depending on the inequality symbol.
 - Determine the roots of the related equation, and then use a number line and test points to determine the intervals that satisfy the inequality.
 - Determine when each of the factors of the quadratic expression is positive, zero, or negative, and then use the results to determine the sign of the product.
 - Consider all cases for the required product of the factors of the quadratic expression to find any x -values that satisfy both factor conditions in each case.
- For inequalities with the symbol \geq or \leq , include the x -intercepts in the solution set.

Check Your Understanding

Practise

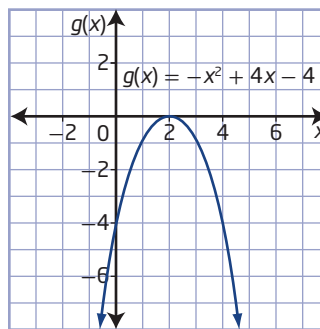
1. Consider the graph of the quadratic function $f(x) = x^2 - 4x + 3$.



What is the solution to

- $x^2 - 4x + 3 \leq 0$?
- $x^2 - 4x + 3 \geq 0$?
- $x^2 - 4x + 3 > 0$?
- $x^2 - 4x + 3 < 0$?

2. Consider the graph of the quadratic function $g(x) = -x^2 + 4x - 4$.



What is the solution to

- $-x^2 + 4x - 4 \leq 0$?
- $-x^2 + 4x - 4 \geq 0$?
- $-x^2 + 4x - 4 > 0$?
- $-x^2 + 4x - 4 < 0$?

3. Is the value of x a solution to the given inequality?

- $x = 4$ for $x^2 - 3x - 10 > 0$
- $x = 1$ for $x^2 + 3x - 4 \geq 0$
- $x = -2$ for $x^2 + 4x + 3 < 0$
- $x = -3$ for $-x^2 - 5x - 4 \leq 0$

4. Use roots and test points to determine the solution to each inequality.
 - a) $x(x + 6) \geq 40$
 - b) $-x^2 - 14x - 24 < 0$
 - c) $6x^2 > 11x + 35$
 - d) $8x + 5 \leq -2x^2$
5. Use sign analysis to determine the solution to each inequality.
 - a) $x^2 + 3x \leq 18$
 - b) $x^2 + 3 \geq -4x$
 - c) $4x^2 - 27x + 18 < 0$
 - d) $-6x \geq x^2 - 16$
6. Use case analysis to determine the solution to each inequality.
 - a) $x^2 - 2x - 15 < 0$
 - b) $x^2 + 13x > -12$
 - c) $-x^2 + 2x + 5 \leq 0$
 - d) $2x^2 \geq 8 - 15x$
7. Use graphing to determine the solution to each inequality.
 - a) $x^2 + 14x + 48 \leq 0$
 - b) $x^2 \geq 3x + 28$
 - c) $-7x^2 + x - 6 \geq 0$
 - d) $4x(x - 1) > 63$
8. Solve each of the following inequalities. Explain your strategy and why you chose it.
 - a) $x^2 - 10x + 16 < 0$
 - b) $12x^2 - 11x - 15 \geq 0$
 - c) $x^2 - 2x - 12 \leq 0$
 - d) $x^2 - 6x + 9 > 0$
9. Solve each inequality.
 - a) $x^2 - 3x + 6 \leq 10x$
 - b) $2x^2 + 12x - 11 > x^2 + 2x + 13$
 - c) $x^2 - 5x < 3x^2 - 18x + 20$
 - d) $-3(x^2 + 4) \leq 3x^2 - 5x - 68$

Apply

10. Each year, Dauphin, Manitoba, hosts the largest ice-fishing contest in Manitoba. Before going on any ice, it is important to know that the ice is thick enough to support the intended load. The solution to the inequality $9h^2 \geq 750$ gives the thickness, h , in centimetres, of ice that will support a vehicle of mass 750 kg.



- a) Solve the inequality to determine the minimum thickness of ice that will safely support the vehicle.
- b) Write a new inequality, in the form $9h^2 \geq \text{mass}$, that you can use to find the ice thickness that will support a mass of 1500 kg.
- c) Solve the inequality you wrote in part b).
- d) Why is the thickness of ice required to support 1500 kg not twice the thickness needed to support 750 kg? Explain.

Did You Know?

Conservation efforts at Dauphin Lake, including habitat enhancement, stocking, and education, have resulted in sustainable fish stocks and better fishing for anglers.

11. Many farmers in Southern Alberta irrigate their crops. A centre-pivot irrigation system spreads water in a circular pattern over a crop.



- a) Suppose that Murray has acquired rights to irrigate up to 63 ha (hectares) of his land. Write an inequality to model the maximum circular area, in square metres, that he can irrigate.
- b) What are the possible radii of circles that Murray can irrigate? Express your answer as an exact value.
- c) Express your answer in part b) to the nearest hundredth of a metre.

Did You Know?

The hectare is a unit of area defined as 10 000 m². It is primarily used as a measurement of land area.

12. Suppose that an engineer determines that she can use the formula $-t^2 + 14 \leq P$ to estimate when the price of carbon fibre will be P dollars per kilogram or less in t years from the present.
- a) When will carbon fibre be available at \$10/kg or less?
 - b) Explain why some of the values of t that satisfy the inequality do not solve the problem.

- c) Write and solve a similar inequality to determine when carbon fibre prices will drop below \$5/kg.



Did You Know?

Carbon fibre is prized for its high strength-to-mass ratio. Prices for carbon fibre were very high when the technology was new, but dropped as manufacturing methods improved.

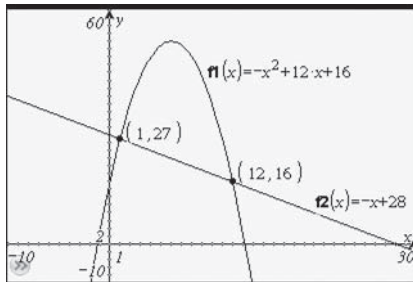
13. One leg of a right triangle is 2 cm longer than the other leg. How long should the shorter leg be to ensure that the area of the triangle is greater than or equal to 4 cm²?

Extend

14. Use your knowledge of the graphs of quadratic functions and the discriminant to investigate the solutions to the quadratic inequality $ax^2 + bx + c \geq 0$.
- a) Describe all cases where all real numbers satisfy the inequality.
 - b) Describe all cases where exactly one real number satisfies the inequality.
 - c) Describe all cases where infinitely many real numbers satisfy the inequality and infinitely many real numbers do not satisfy the inequality.
15. For each of the following, give an inequality that has the given solution.
- a) $-2 \leq x \leq 7$
 - b) $x < 1$ or $x > 10$
 - c) $\frac{5}{3} \leq x \leq 6$
 - d) $x < -\frac{3}{4}$ or $x > -\frac{1}{5}$
 - e) $x \leq -3 - \sqrt{7}$ or $x \geq -3 + \sqrt{7}$
 - f) $x \in \mathbb{R}$
 - g) no solution

16. Solve $|x^2 - 4| \geq 2$.

17. The graph shows the solution to the inequality $-x^2 + 12x + 16 \geq -x + 28$.



- Why is $1 \leq x \leq 12$ the solution to the inequality?
- Rearrange the inequality so that it has the form $q(x) \geq 0$ for a quadratic $q(x)$.
- Solve the inequality you determined in part b).
- How are the solutions to parts a) and c) related? Explain.

Create Connections

18. In Example 3, the first step in the solution was to rearrange the inequality $2x^2 - 7x > 12$ into $2x^2 - 7x - 12 > 0$. Which solution methods require this first step and which do not? Show the work that supports your conclusions.

19. Compare and contrast the methods of graphing, roots and test points, sign analysis, and case analysis. Explain which of the methods you prefer to use and why.

20. Devan needs to solve $x^2 + 5x + 4 \leq -2$. His solutions are shown.

Devan's solution:

Begin by rewriting the inequality: $x^2 + 5x + 6 \geq 0$

Factor the left side: $(x + 2)(x + 3) \geq 0$. Then, consider two cases:

Case 1:

$$(x + 2) \leq 0 \text{ and } (x + 3) \leq 0$$

Then, $x \leq -2$ and $x \leq -3$, so the solution is $x \leq -3$.

Case 2:

$$(x + 2) \geq 0 \text{ and } (x + 3) \geq 0$$

Then, $x \geq -2$ and $x \geq -3$, so the solution is $x \geq -2$.

From the two cases, the solution to the inequality is $x \leq -3$ or $x \geq -2$.

- Decide whether his solution is correct. Justify your answer.
- Use a different method to confirm the correct answer to the inequality.

Project Corner

Financial Considerations

- Currently, the methods of nanotechnology in several fields are very expensive. However, as is often the case, it is expected that as technology improves, the costs will decrease. Nanotechnology seems to have the potential to decrease costs in the future. It also promises greater flexibility and greater precision in the manufacturing of goods.
- What changes in manufacturing might help lower the cost of nanotechnology?



Quadratic Inequalities in Two Variables

Focus on...

- explaining how to use test points to find the solution to an inequality
- explaining when a solid or a dashed line should be used in the solution to an inequality
- sketching, with or without technology, the graph of a quadratic inequality
- solving a problem that involves a quadratic inequality

An arch is a common way to span a doorway or window. A parabolic arch is the strongest possible arch because the arch is self-supporting. This is because the shape of the arch causes the force of gravity to hold the arch together instead of pulling the arch apart.

There are many things to consider when designing an arch. One important decision is the height of the space below the arch. To ensure that the arch is functional, the designer can set up and solve a quadratic inequality in two variables. Quadratic inequalities are applied in physics, engineering, architecture, and many other fields.



Investigate Quadratic Inequalities in Two Variables

Materials

- grid paper
- coloured pens, pencils, or markers

1. Sketch the graph of the function $y = x^2$.
2. a) Label four points on the graph and copy and complete the table for these points. One has been done for you.

x	y	Satisfies the Equation $y = x^2$?
3	9	$9 = 3^2$ Yes

- b) What can you conclude about the points that lie on the parabola?

3. The parabola that you graphed in step 1 divides the Cartesian plane into two regions, one above and one below the parabola.
 - a) In which of these regions do you think the solution set for $y < x^2$ lies?
 - b) Plot four points in this region of the plane and create a table similar to the one in step 2, using the heading “Satisfies the Inequality $y < x^2$?” for the last column.
4. Were you correct in your thinking of which region the solution set for $y < x^2$ lies in? How do you know?
5. Shade the region containing the solution set for the inequality $y < x^2$.
6. a) In which region does the solution set for $y > x^2$ lie?
 - b) Plot four points in this region of the plane and create a table similar to the one in step 2, using the heading “Satisfies the Inequality $y > x^2$?” for the last column.
7. Did the table verify the region you chose for the set of points that satisfy $y > x^2$?
8. Shade the region containing the solution set for the inequality $y > x^2$.

Reflect and Respond

9. Why is a shaded region used to represent the solution sets in steps 5 and 8?
10. Make a conjecture about how you can identify the solution region of the graph of a quadratic inequality.
11. Under what conditions would the graph of the function be part of the solution region for a quadratic inequality?

Link the Ideas

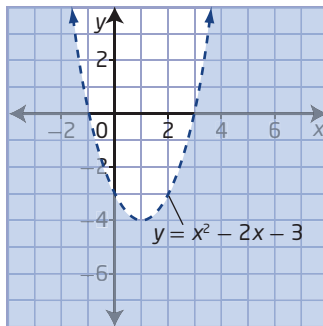
You can express a quadratic inequality in two variables in one of the following four forms:

- $y < ax^2 + bx + c$
- $y \leq ax^2 + bx + c$
- $y > ax^2 + bx + c$
- $y \geq ax^2 + bx + c$

where a , b , and c are real numbers and $a \neq 0$.

A quadratic inequality in two variables represents a region of the Cartesian plane with a parabola as the boundary. The graph of a quadratic inequality is the set of points (x, y) that are solutions to the inequality.

Consider the graph of $y < x^2 - 2x - 3$.



The boundary is the related parabola $y = x^2 - 2x - 3$. Since the inequality symbol is $<$, points on the boundary line are not included in the solution region, so the curve is drawn as a dashed line.

To determine which region is the solution region, choose a test point from either above or below the parabola. If the coordinates of the test point satisfy the inequality, then shade the region containing the test point. If the coordinates do not satisfy the inequality, then shade the region that does not contain the test point.

Try $(0, 0)$, which is above the parabola.

Left Side	Right Side
y	$x^2 - 2x - 3$
$= 0$	$= 0^2 - 2(0) - 3$
	$= -3$

Left Side \nless Right Side

The point $(0, 0)$ does not satisfy the inequality. Thus, shade the region below the parabola.

Example 1

Graph a Quadratic Inequality in Two Variables With $a < 0$

- Graph $y < -2(x - 3)^2 + 1$.
- Determine if the point $(2, -4)$ is a solution to the inequality.

Solution

- Graph the related parabola $y = -2(x - 3)^2 + 1$. Since the inequality symbol is $<$, draw the parabola as a dashed line, indicating that it is not part of the solution.

Use test points to decide which of the two regions contains the solutions to the inequality.

How can you use the values of a , p , and q to graph the parabola?

Choose $(0, 0)$ and $(3, -3)$.

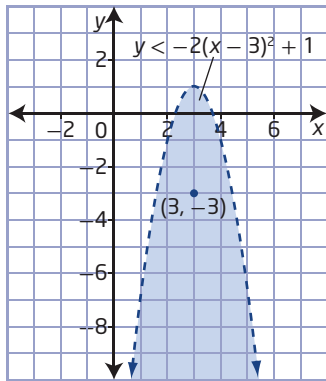
Left Side	Right Side
y	$-2(x - 3)^2 + 1$
$= 0$	$= -2(0 - 3)^2 + 1$
	$= -18 + 1$
	$= -17$

Left Side \nless Right Side

Left Side	Right Side
y	$-2(x - 3)^2 + 1$
$= -3$	$= -2(3 - 3)^2 + 1$
	$= 0 + 1$
	$= 1$

Left Side $<$ Right Side

The point $(3, -3)$ satisfies the inequality, so shade the region below the parabola.



- b)** From the graph of $y < -2(x - 3)^2 + 1$, the point $(2, -4)$ is in the solution region. It is part of the solution to the inequality $y < -2(x - 3)^2 + 1$. Verify this by substituting in the inequality.

Left Side	Right Side
y	$-2(x - 3)^2 + 1$
$= -4$	$= -2(2 - 3)^2 + 1$
	$= -2 + 1$
	$= -1$

Left Side $<$ Right Side

Your Turn

- a)** Graph $y > (x - 4)^2 - 2$.
b) Determine if the point $(2, 1)$ is a solution to the inequality.

Example 2

Graph a Quadratic Inequality in Two Variables With $a > 0$

Graph $y \geq x^2 - 4x - 5$.

Solution

Graph the related parabola $y = x^2 - 4x - 5$. Since the inequality symbol is \geq , points on the parabola are included in the solution. Draw the parabola using a solid line.

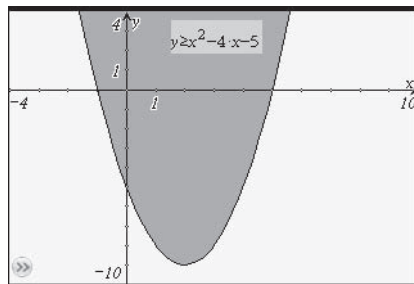
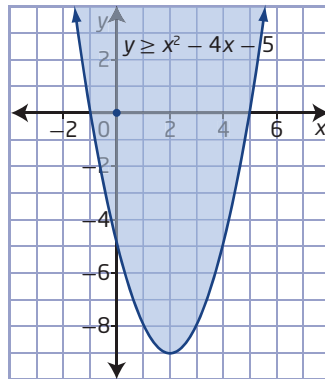
Use a test point from one region to decide whether that region contains the solutions to the inequality.

Choose $(0, 0)$.

Left Side	Right Side
y	$x^2 - 4x - 5$
$= 0$	$= 0^2 - 4(0) - 5$
	$= 0 - 0 - 5$
	$= -5$

$$\text{Left Side} \geq \text{Right Side}$$

The point $(0, 0)$ satisfies the inequality, so shade the region above the parabola.



Your Turn

Graph $y \leq -x^2 + 2x + 4$.

Example 3

Determine the Quadratic Inequality That Defines a Solution Region

You can use a parabolic reflector to focus sound, light, or radio waves to a single point. A parabolic microphone has a parabolic reflector attached that directs incoming sounds to the microphone. René, a journalist, is using a parabolic microphone as he covers the Francophone Summer Festival of Vancouver. Describe the region that René can cover with his microphone if the reflector has a width of 50 cm and a maximum depth of 15 cm.

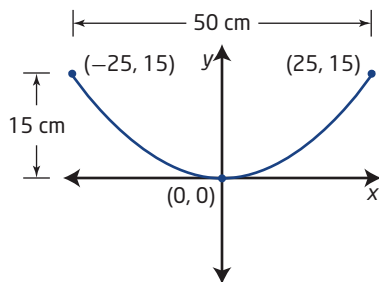
Solution

Method 1: Describe Graphically

Draw a diagram and label it with the given information.

Let the origin represent the vertex of the parabolic reflector.

Let x and y represent the horizontal and vertical distances, in centimetres, from the low point in the centre of the parabolic reflector.



From the graph, the region covered lies between -25 cm to $+25$ cm because of the width of the microphone.

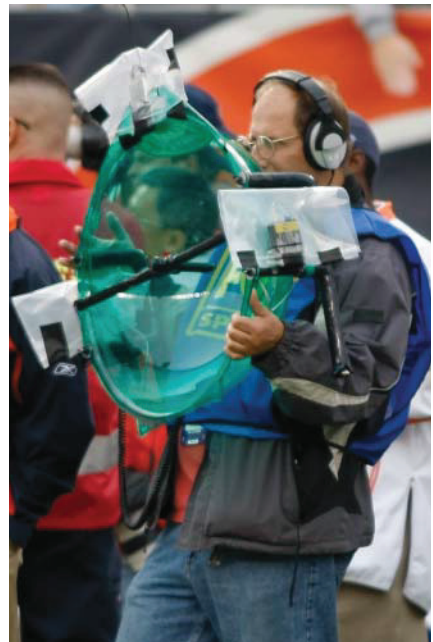
Method 2: Describe Algebraically

You can write a quadratic function to represent a parabola if you know the coordinates of the vertex and one other point.

Since the vertex is $(0, 0)$, the function is of the form $y = ax^2$.

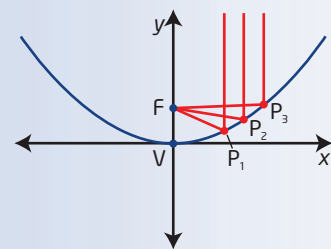
Substitute the coordinates of the top of one edge of the parabolic reflector, $(25, 15)$, and solve to find $a = \frac{3}{125}$.

$$y = \frac{3}{125}x^2$$

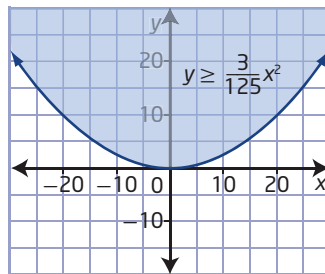


Did You Know?

A parabolic reflector can be used to collect and concentrate energy entering the reflector. A parabolic reflector causes incoming rays in the form of light, sound, or radio waves, that are parallel to the axis of the dish, to be reflected to a central point called the focus. Similarly, energy radiating from the focus to the dish can be transmitted outward in a beam that is parallel to the axis of the dish.



The microphone picks up sound from the space above the graph of the quadratic function. So, shade the region above the parabola.



Why is the reflector represented by a solid curve rather than a broken curve?

However, the maximum scope is from -25 to $+25$ because of the width of the microphone. So, the domain of the region covered by the microphone is restricted to $\{x \mid -25 \leq x \leq 25, x \in \mathbb{R}\}$.

Use a test point from the solution region to verify the inequality symbol.

Choose the point $(5, 5)$.

Left Side	Right Side
y	$\frac{3}{125}x^2$
$= 5$	$= \frac{3}{125}(5)^2$
	$= \frac{3}{5}$

$$\text{Left Side} \geq \text{Right Side}$$

The region covered by the microphone can be described by the quadratic inequality $y \geq \frac{3}{125}x^2$, where $-25 \leq x \leq 25$.

Your Turn

A satellite dish is 60 cm in diameter and 20 cm deep. The dish has a parabolic cross-section. Locate the vertex of the parabolic cross-section at the origin, and sketch the parabola that represents the dish. Determine an inequality that shows the region from which the dish can receive a signal.



Your Turn

Sports climbers use a rope that is longer and supports less mass than manila rope. The rope can safely support a mass, M , in pounds, modelled by the inequality $M \leq 1240(d - 2)^2$, where d is the diameter of the rope, in inches. Graph the inequality to examine how the mass that the rope supports is related to the diameter of the rope.

Key Ideas

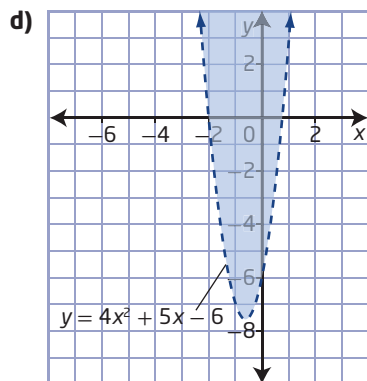
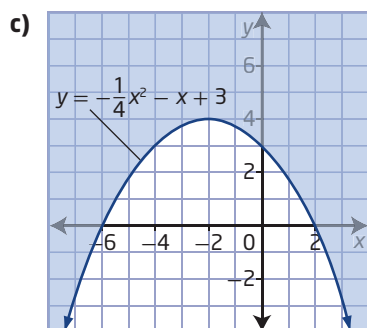
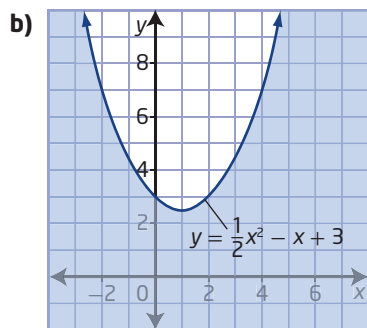
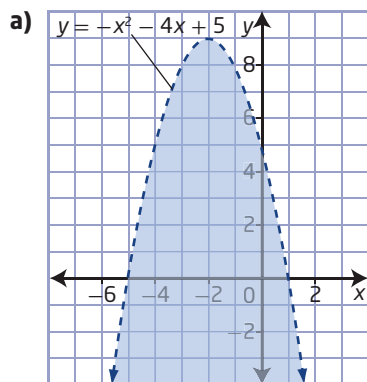
- A quadratic inequality in two variables represents a region of the Cartesian plane containing the set of points that are solutions to the inequality.
- The graph of the related quadratic function is the boundary that divides the plane into two regions.
 - When the inequality symbol is \leq or \geq , include the points on the boundary in the solution region and draw the boundary as a solid line.
 - When the inequality symbol is $<$ or $>$, do not include the points on the boundary in the solution region and draw the boundary as a dashed line.
- Use a test point to determine the region that contains the solutions to the inequality.

Check Your Understanding

Practise

- Which of the ordered pairs are solutions to the inequality?
 - $y < x^2 + 3$,
 $\{(2, 6), (4, 20), (-1, 3), (-3, 12)\}$
 - $y \leq -x^2 + 3x - 4$,
 $\{(2, -2), (4, -1), (0, -6), (-2, -15)\}$
 - $y > 2x^2 + 3x + 6$,
 $\{(-3, 5), (0, -6), (2, 10), (5, 40)\}$
 - $y \geq -\frac{1}{2}x^2 - x + 5$,
 $\{(-4, 2), (-1, 5), (1, 3.5), (3, 2.5)\}$
- Which of the ordered pairs are *not* solutions to the inequality?
 - $y \geq 2(x - 1)^2 + 1$,
 $\{(0, 1), (1, 0), (3, 6), (-2, 15)\}$
 - $y > -(x + 2)^2 - 3$,
 $\{(-3, 1), (-2, -3), (0, -8), (1, 2)\}$
 - $y \leq \frac{1}{2}(x - 4)^2 + 5$,
 $\{(0, 4), (3, 1), (4, 5), (2, 9)\}$
 - $y < -\frac{2}{3}(x + 3)^2 - 2$,
 $\{(-2, 2), (-1, -5), (-3, -2), (0, -10)\}$

3. Write an inequality to describe each graph, given the function defining the boundary parabola.



4. Graph each quadratic inequality using transformations to sketch the boundary parabola.

a) $y \geq 2(x + 3)^2 + 4$

b) $y > -\frac{1}{2}(x - 4)^2 - 1$

c) $y < 3(x + 1)^2 + 5$

d) $y \leq \frac{1}{4}(x - 7)^2 - 2$

5. Graph each quadratic inequality using points and symmetry to sketch the boundary parabola.

a) $y < -2(x - 1)^2 - 5$

b) $y > (x + 6)^2 + 1$

c) $y \geq \frac{2}{3}(x - 8)^2$

d) $y \leq \frac{1}{2}(x + 7)^2 - 4$

6. Graph each quadratic inequality.

a) $y \leq x^2 + x - 6$

b) $y > x^2 - 5x + 4$

c) $y \geq x^2 - 6x - 16$

d) $y < x^2 + 8x + 16$

7. Graph each inequality using graphing technology.

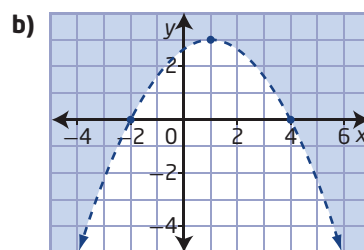
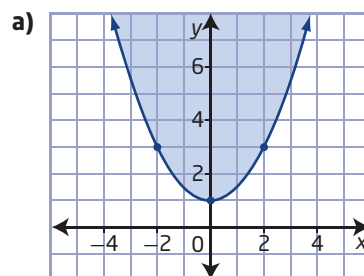
a) $y < 3x^2 + 13x + 10$

b) $y \geq -x^2 + 4x + 7$

c) $y \leq x^2 + 6$

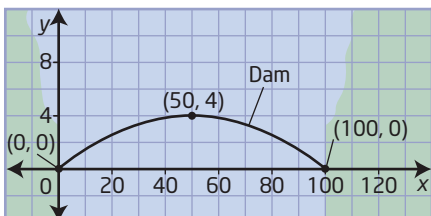
d) $y > -2x^2 + 5x - 8$

8. Write an inequality to describe each graph.



Apply

9. When a dam is built across a river, it is often constructed in the shape of a parabola. A parabola is used so that the force that the river exerts on the dam helps hold the dam together. Suppose a dam is to be built as shown in the diagram.



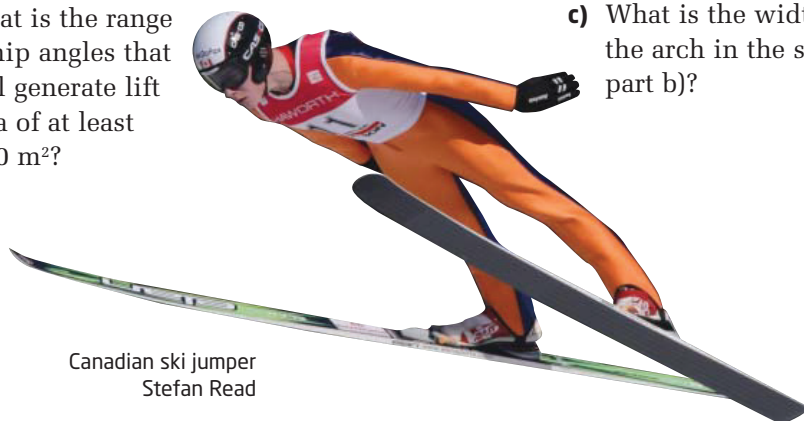
- What is the quadratic function that models the parabolic arch of the dam?
- Write the inequality that approximates the region below the parabolic arch of the dam.

Did You Know?

The Mica Dam, which spans the Columbia River near Revelstoke, British Columbia, is a parabolic dam that provides hydroelectric power to Canada and parts of the United States.

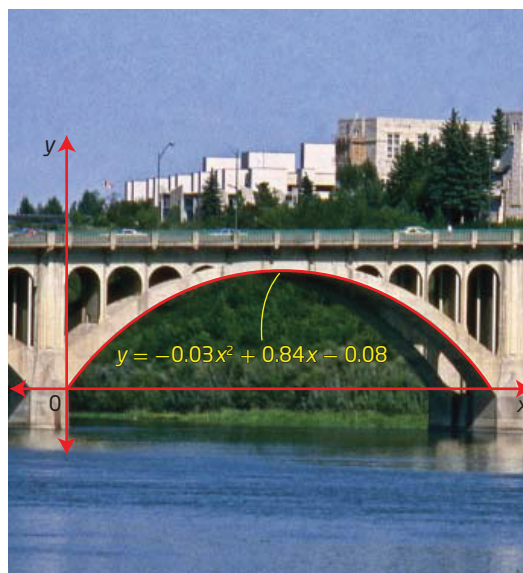
10. In order to get the longest possible jump, ski jumpers need to have as much lift area, L , in square metres, as possible while in the air. One of the many variables that influences the amount of lift area is the hip angle, a , in degrees, of the skier. The relationship between the two is given by $L \geq -0.000\ 125a^2 + 0.040a - 2.442$.

- Graph the quadratic inequality.
- What is the range of hip angles that will generate lift area of at least 0.50 m^2 ?



Canadian ski jumper
Stefan Read

11. The University Bridge in Saskatoon is supported by several parabolic arches. The diagram shows how a Cartesian plane can be applied to one arch of the bridge. The function $y = -0.03x^2 + 0.84x - 0.08$ approximates the curve of the arch, where x represents the horizontal distance from the bottom left edge and y represents the height above where the arch meets the vertical pier, both in metres.



- Write the inequality that approximates the possible water levels below the parabolic arch of the bridge.
- Suppose that the normal water level of the river is at most 0.2 m high, relative to the base of the arch. Write and solve an inequality to represent the normal river level below the arch.
- What is the width of the river under the arch in the situation described in part b)?

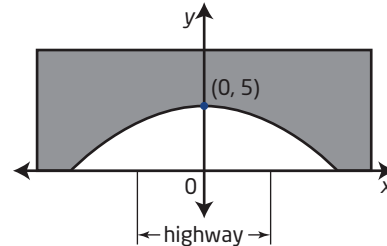
12. In order to conduct microgravity research, the Canadian Space Agency uses a Falcon 20 jet that flies a parabolic path. As the jet nears the vertex of the parabola, the passengers in the jet experience nearly zero gravity that lasts for a short period of time. The function $h = -2.944t^2 + 191.360t + 6950.400$ models the flight of a jet on a parabolic path for time, t , in seconds, since weightlessness has been achieved and height, h , in metres.



Canadian Space Agency astronauts David Saint-Jacques and Jeremy Hansen experience microgravity during a parabolic flight as part of basic training.

- The passengers begin to experience weightlessness when the jet climbs above 9600 m. Write an inequality to represent this information.
- Determine the time period for which the jet is above 9600 m.
- For how long does the microgravity exist on the flight?

13. A highway goes under a bridge formed by a parabolic arch, as shown. The highest point of the arch is 5 m high. The road is 10 m wide, and the minimum height of the bridge over the road is 4 m.



- Determine the quadratic function that models the parabolic arch of the bridge.
- What is the inequality that represents the space under the bridge in quadrants I and II?

Extend

14. Tavia has been adding advertisements to her Web site. Initially her revenue increased with each additional ad she included on her site. However, as she kept increasing the number of ads, her revenue began to drop. She kept track of her data as shown.

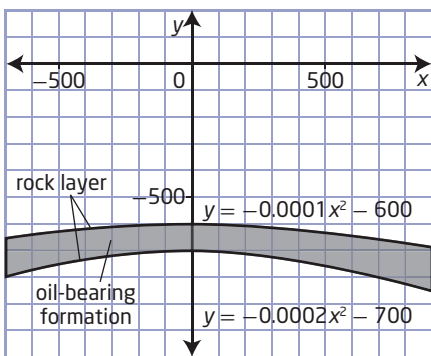
Number of Ads	0	10	15
Revenue (\$)	0	100	75

- Determine the quadratic inequality that models Tavia's revenue.
- How many ads can Tavia include on her Web site to earn revenue of at least \$50?

Did You Know?

The law of diminishing returns is a principle in economics. The law states the surprising result that when you continually increase the quantity of one input, you will eventually see a decrease in the output.

15. Oil is often recovered from a formation bounded by layers of rock that form a parabolic shape. Suppose a geologist has discovered such an oil-bearing formation. The quadratic functions that model the rock layers are $y = -0.0001x^2 - 600$ and $y = -0.0002x^2 - 700$, where x represents the horizontal distance from the centre of the formation and y represents the depth below ground level, both in metres. Write the inequality that describes the oil-bearing formation.



Create Connections

16. To raise money, the student council sells candy-grams each year. From past experience, they expect to sell 400 candy-grams at a price of \$4 each. They have also learned from experience that each \$0.50 increase in the price causes a drop in sales of 20 candy-grams.
- Write an equality that models this situation. Define your variables.
 - Suppose the student council needs revenue of at least \$1800. Solve an inequality to find all the possible prices that will achieve the fundraising goal.
 - Show how your solution would change if the student council needed to raise \$1600 or more.
17. An environmentalist has been studying the methane produced by an inactive landfill. To approximate the methane produced, p , as a percent of peak output compared to time, t , in years, after the year 2000, he uses the inequality $p \leq 0.24t^2 - 8.1t + 74$.



- For what time period is methane production below 10% of the peak production?
 - Graph the inequality used by the environmentalist. Explain why only a portion of the graph is a reasonable model for the methane output of the landfill. Which part of the graph would the environmentalist use?
 - Modify your answer to part a) to reflect your answer in part b).
 - Explain how the environmentalist can use the concept of domain to make modelling the situation with the quadratic inequality more reasonable.
18. Look back at your work in Unit 2, where you learned about quadratic functions. Working with a partner, identify the concepts and skills you learned in that unit that have helped you to understand the concepts in this unit. Decide which concept from Unit 2 was most important to your understanding in Unit 4. Find another team that chose a different concept as the most important. Set up a debate, with each team defending its choice of most important concept.

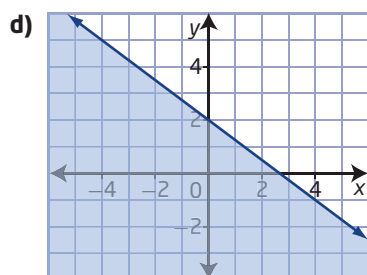
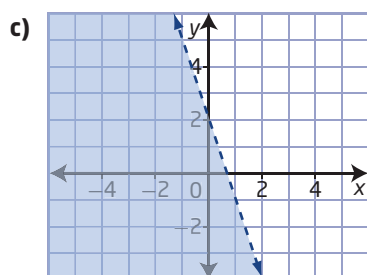
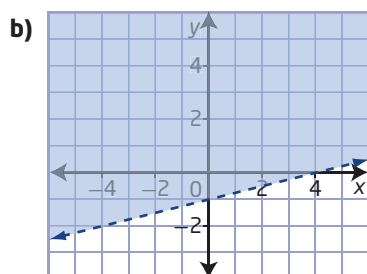
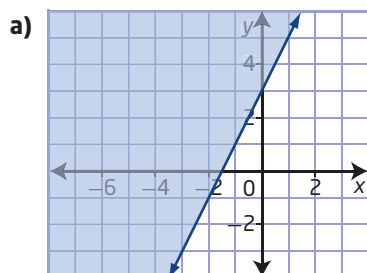
Chapter 9 Review

9.1 Linear Inequalities in Two Variables, pages 464–475

1. Graph each inequality without using technology.

a) $y \leq 3x - 5$
b) $y > -\frac{3}{4}x + 2$
c) $3x - y \geq 6$
d) $4x + 2y \leq 8$
e) $10x - 4y + 3 < 11$

2. Determine the inequality that corresponds to each of the following graphs.



3. Graph each inequality using technology.

a) $4x + 5y > 22$
b) $10x - 4y + 52 \geq 0$
c) $-3.2x + 1.1y < 8$
d) $12.4x + 4.4y > 16.5$
e) $\frac{3}{4}x \leq 9y$

4. Janelle has a budget of \$120 for entertainment each month. She usually spends the money on a combination of movies and meals. Movie admission, with popcorn, is \$15, while a meal costs \$10.

- a) Write an inequality to represent the number of movies and meals that Janelle can afford with her entertainment budget.
b) Graph the solution.
c) Interpret your solution. Explain how the solution to the inequality relates to Janelle's situation.

5. Jodi is paid by commission as a salesperson. She earns 5% commission for each laptop computer she sells and 8% commission for each DVD player she sells. Suppose that the average price of a laptop is \$600 and the average price of a DVD player is \$200.

- a) What is the average amount Jodi earns for selling each item?
b) Jodi wants to earn a minimum commission this month of \$1000. Write an inequality to represent this situation.
c) Graph the inequality. Interpret your results in the context of Jodi's earnings.

9.2 Quadratic Inequalities in One Variable, pages 476–487

6. Choose a strategy to solve each inequality. Explain your strategy and why you chose it.

a) $x^2 - 2x - 63 > 0$

b) $2x^2 - 7x - 30 \geq 0$

c) $x^2 + 8x - 48 < 0$

d) $x^2 - 6x + 4 \geq 0$

7. Solve each inequality.

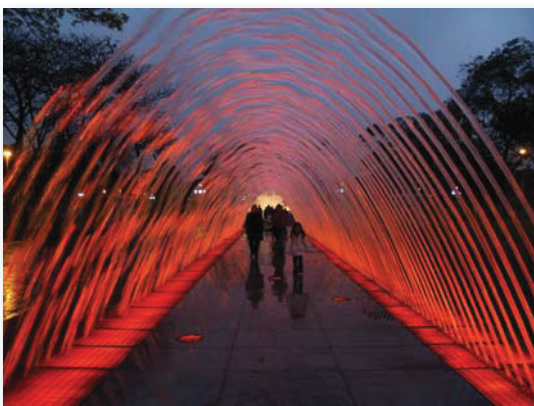
a) $x(6x + 5) \leq 4$

b) $4x^2 < 10x - 1$

c) $x^2 \leq 4(x + 8)$

d) $5x^2 \geq 4 - 12x$

8. A decorative fountain shoots water in a parabolic path over a pathway. To determine the location of the pathway, the designer must solve the inequality $-\frac{3}{4}x^2 + 3x \leq 2$, where x is the horizontal distance from the water source, in metres.



- a) Solve the inequality.
b) Interpret the solution to the inequality for the fountain designer.
9. A rectangular storage shed is to be built so that its length is twice its width. If the maximum area of the floor of the shed is 18 m^2 , what are the possible dimensions of the shed?

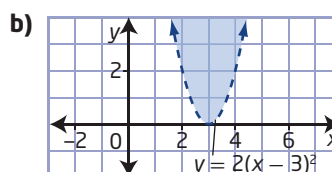
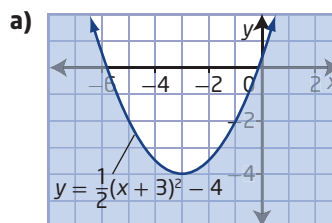
10. David has learned that the light from the headlights reaches about 100 m ahead of the car he is driving. If v represents David's speed, in kilometres per hour, then the inequality $0.007v^2 + 0.22v \leq 100$ gives the speeds at which David can stop his vehicle in 100 m or less.



- a) What is the maximum speed at which David can travel and safely stop his vehicle in the 100-m distance?
b) Modify the inequality so that it gives the speeds at which a vehicle can stop in 50 m or less.
c) Solve the inequality you wrote in part b). Explain why your answer is not half the value of your answer for part a).

9.3 Quadratic Inequalities in Two Variables, pages 488–500

11. Write an inequality to describe each graph, given the function defining the boundary parabola.



12. Graph each quadratic inequality.

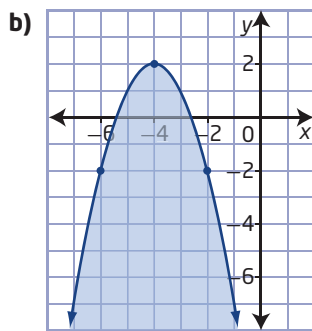
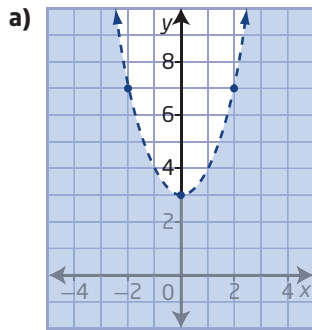
a) $y < x^2 + 2x - 15$

b) $y \geq -x^2 + 4$

c) $y > 6x^2 + x - 12$

d) $y \leq (x - 1)^2 - 6$

13. Write an inequality to describe each graph.



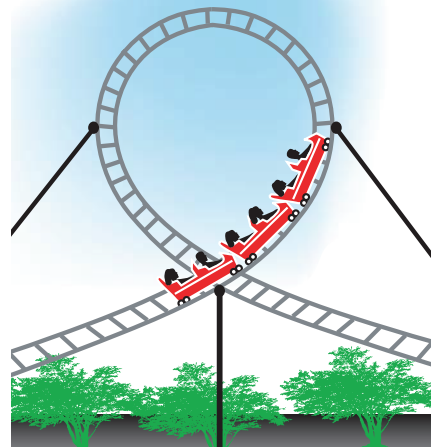
14. You can model the maximum Saskatchewan wheat production for the years 1975 to 1995 with the function $y = 0.003t^2 - 0.052t + 1.986$, where t is the time, in years, after 1975 and y is the yield, in tonnes per hectare.

- a) Write and graph an inequality to model the potential wheat production during this period.
- b) Write and solve an inequality to represent the years in which production is at most 2 t/ha.

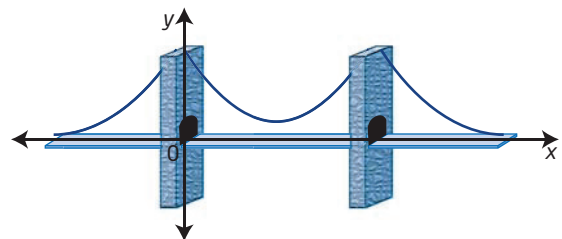
Did You Know?

Saskatchewan has 44% of Canada's total cultivated farmland. Over 10% of the world's total exported wheat comes from this province.

15. An engineer is designing a roller coaster for an amusement park. The speed at which the roller coaster can safely complete a vertical loop is approximated by $v^2 \geq 10r$, where v is the speed, in metres per second, of the roller coaster and r is the radius, in metres, of the loop.



- a) Graph the inequality to examine how the radius of the loop is related to the speed of the roller coaster.
- b) A vertical loop of the roller coaster has a radius of 16 m. What are the possible safe speeds for this vertical loop?
16. The function $y = \frac{1}{20}x^2 - 4x + 90$ models the cable that supports a suspension bridge, where x is the horizontal distance, in metres, from the base of the first support and y is the height, in metres, of the cable above the bridge deck.



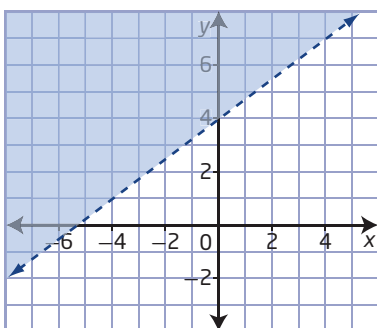
- a) Write an inequality to determine the points for which the height of the cable is at least 20 m.
- b) Solve the inequality. What does the solution represent?

Chapter 9 Practice Test

Multiple Choice

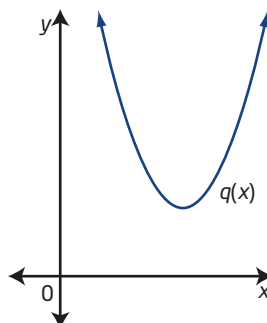
For #1 to #5, choose the best answer.

- An inequality that is equivalent to $3x - 6y < 12$ is
 - $y < \frac{1}{2}x - 2$
 - $y > \frac{1}{2}x - 2$
 - $y < 2x - 2$
 - $y > 2x - 2$
- What linear inequality does the graph show?

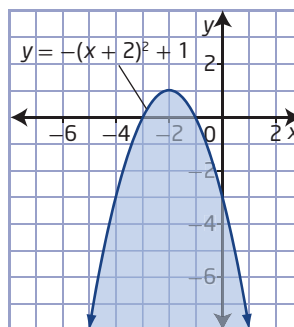


- $y > \frac{3}{4}x + 4$
 - $y \geq \frac{3}{4}x + 4$
 - $y < \frac{4}{3}x + 4$
 - $y \leq \frac{4}{3}x + 4$
- What is the solution set for the quadratic inequality $6x^2 - 7x - 20 < 0$?
 - $\left\{x \mid x \leq -\frac{4}{3} \text{ or } x \geq \frac{5}{2}, x \in \mathbb{R}\right\}$
 - $\left\{x \mid -\frac{4}{3} \leq x \leq \frac{5}{2}, x \in \mathbb{R}\right\}$
 - $\left\{x \mid -\frac{4}{3} < x < \frac{5}{2}, x \in \mathbb{R}\right\}$
 - $\left\{x \mid x < -\frac{4}{3} \text{ or } x > \frac{5}{2}, x \in \mathbb{R}\right\}$

- For the quadratic function $q(x)$ shown in the graph, which of the following is true?



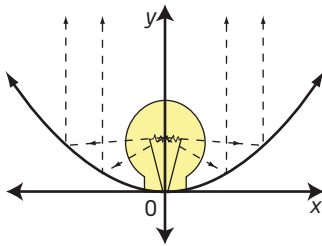
- There are no solutions to $q(x) > 0$.
 - All real numbers are solutions to $q(x) \geq 0$.
 - All real numbers are solutions to $q(x) \leq 0$.
 - All positive real numbers are solutions to $q(x) < 0$.
- What quadratic inequality does the graph show?



- $y < -(x + 2)^2 + 1$
- $y \geq -(x + 2)^2 + 1$
- $y \leq -(x + 2)^2 + 1$
- $y > -(x + 2)^2 + 1$

Short Answer

6. Graph $8x \geq 2(y - 5)$.
7. Solve $12x^2 < 7x + 10$.
8. Graph $y > (x - 5)^2 + 4$.
9. Stage lights often have parabolic reflectors to make it possible to focus the beam of light, as indicated by the diagram.



Suppose the reflector in a stage light is represented by the function $y = 0.02x^2$. What inequality can you use to model the region illuminated by the light?

10. While on vacation, Ben has \$300 to spend on recreation. Scuba diving costs \$25/h and sea kayaking costs \$20/h. What are all the possible ways that Ben can budget his recreation money?



Extended Response

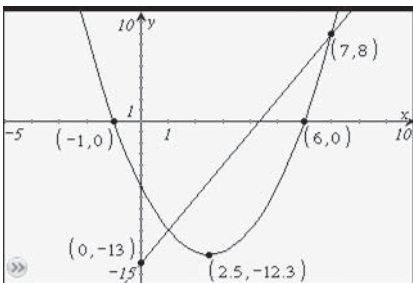
11. Malik sells his artwork for different prices depending on the type of work. Pen and ink sketches sell for \$50, and watercolours sell for \$80.
 - a) Malik needs an income of at least \$1200 per month. Write an inequality to model this situation.
 - b) Graph the inequality. List three different ordered pairs in the solution.
 - c) Suppose Malik now needs at least \$2400 per month. Write an inequality to represent this new situation. Predict how the answer to this inequality will be related to your answer in part b).
 - d) Solve the new inequality from part c) to check your prediction.
12. Let $f(x)$ represent a quadratic function.
 - a) State a quadratic function for which the solution set to $f(x) \leq 0$ is $\{x \mid -3 \leq x \leq 5, x \in \mathbb{R}\}$. Justify your answer.
 - b) Describe all quadratics for which solutions to $f(x) \leq 0$ are of the form $m \leq x \leq n$ for some real numbers m and n .
 - c) For your answer in part b), explain whether it is more convenient to express quadratic functions in the form $f(x) = ax^2 + bx + c$ or $f(x) = a(x - p)^2 + q$, and why.
13. The normal systolic blood pressure, p , in millimetres of mercury (mmHg), for a woman a years old is given by $p = 0.01a^2 + 0.05a + 107$.
 - a) Write an inequality that expresses the ages for which you expect systolic blood pressure to be less than 120 mmHg.
 - b) Solve the inequality you wrote in part a).
 - c) Are all of the solutions to your inequality realistic answers for this problem? Explain why or why not.

Unit 4 Test

Multiple Choice

For #1 to #9, choose the best answer.

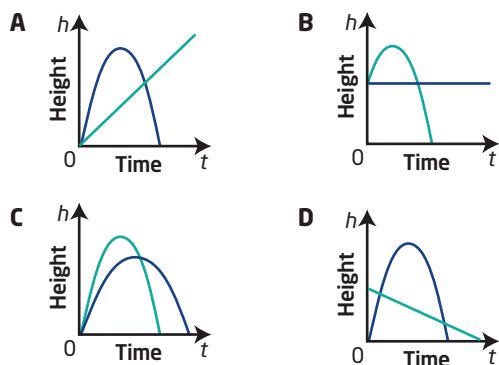
1. Which of the following ordered pairs is a solution to the system of linear-quadratic equations?



- A (2.5, -12.3) B (6, 0)
C (7, 8) D (0, -13)
2. Kelowna, British Columbia, is one of the many places in western Canada with bicycle motocross (BMX) race tracks for teens.



Which graph models the height versus time of two of the racers travelling over one of the jumps?



3. The ordered pairs (1, 3) and (-3, -5) are the solutions to which system of linear-quadratic equations?

- A $y = 3x + 5$
 $y = x^2 - 2x - 1$
B $y = 2x + 1$
 $y = x^2 + 4x - 2$
C $y = x + 2$
 $y = x^2 + 2$
D $y = 4x - 1$
 $y = x^2 - 3x + 5$

4. How many solutions are possible for the following system of quadratic-quadratic equations?

$$y - 5 = 2(x + 1)^2$$

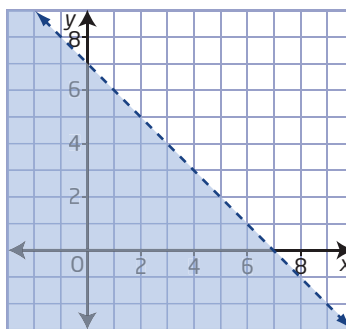
$$y - 5 = -2(x + 1)^2$$

- A zero
B one
C two
D an infinite number

5. Which point cannot be used as a test point to determine the solution region for $4x - y \leq 5$?

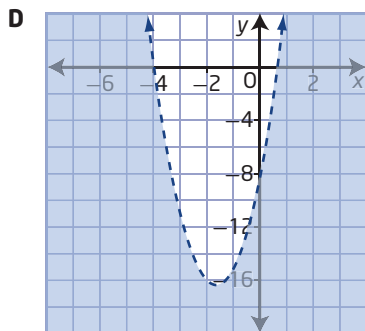
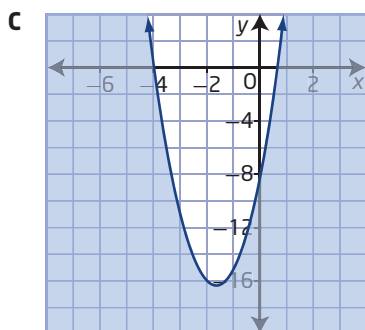
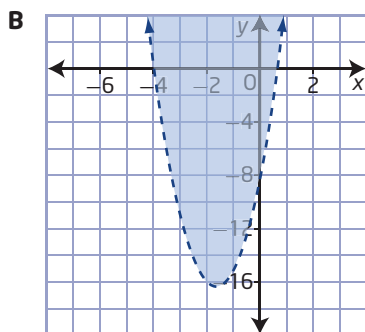
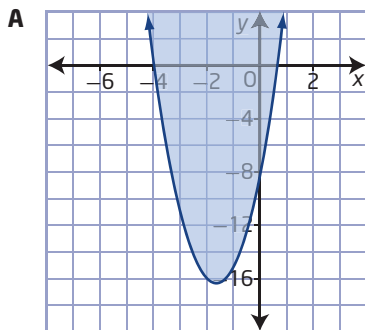
- A (-1, 1) B (2, 5)
C (3, 1) D (2, 3)

6. Which linear inequality does the graph show?



- A $y \leq -x + 7$
B $y \geq -x + 7$
C $y > -x + 7$
D $y < -x + 7$

7. Which graph represents the quadratic inequality $y \geq 3x^2 + 10x - 8$?



8. Determine the solution(s), to the nearest tenth, for the system of quadratic-quadratic equations.

$$y = -\frac{2}{3}x^2 + 2x + 3$$

$$y = x^2 - 4x + 5$$

- A** (3.2, 2.5)
B (3.2, 2.5) and (0.4, 3.7)
C (0.4, 2.5) and (2.5, 3.7)
D (0.4, 3.2)
9. What is the solution set for the quadratic inequality $-3x^2 + x + 11 < 1$?

- A** $\{x \mid x < -\frac{5}{3} \text{ or } x > 2, x \in \mathbb{R}\}$
B $\{x \mid x < -\frac{5}{3} \text{ or } x \geq 2, x \in \mathbb{R}\}$
C $\{x \mid -\frac{5}{3} < x < 2, x \in \mathbb{R}\}$
D $\{x \mid -\frac{5}{3} \leq x \leq 2, x \in \mathbb{R}\}$

Numerical Response

Copy and complete the statements in #10 to #12.

10. One of the solutions for the system of linear-quadratic equations $y = x^2 - 4x - 2$ and $y = x - 2$ is represented by the ordered pair $(a, 3)$, where the value of a is ■.
11. The solution of the system of quadratic-quadratic equations represented by $y = x^2 - 4x + 6$ and $y = -x^2 + 6x - 6$ with the greater coordinates is of the form (a, a) , where the value of a is ■.
12. On a forward somersault dive, Laurie's height, h , in metres, above the water t seconds after she leaves the diving board is approximately modelled by $h(t) = -5t^2 + 5t + 4$. The length of time that Laurie is above 4 m is ■.

Written Response

- 13.** Professional golfers, such as Canadian Mike Weir, make putting look easy to spectators. New technology used on a television sports channel analyses the greens conditions and predicts the path of the golf ball that the golfer should putt to put the ball in the hole. Suppose the straight line from the ball to the hole is represented by the equation $y = 2x$ and the predicted path of the ball is modelled by the equation $y = \frac{1}{4}x^2 + \frac{3}{2}x$.



- a)** Algebraically determine the solution to the system of linear-quadratic equations.
- b)** Interpret the points of intersection in this context.
- 14.** Two quadratic functions, $f(x) = x^2 - 6x + 5$ and $g(x)$, intersect at the points $(2, -3)$ and $(7, 12)$. The graph of $g(x)$ is congruent to the graph of $f(x)$ but opens downward. Determine the equation of $g(x)$ in the form $g(x) = a(x - p)^2 + q$.
- 15.** Algebraically determine the solutions to the system of quadratic-quadratic equations. Verify your solutions.
- $$4x^2 + 8x + 9 - y = 5$$
- $$3x^2 - x + 1 = y + x + 6$$
- 16.** Dolores solved the inequality $3x^2 - 5x - 10 > 2$ using roots and test points. Her solution is shown.
- $$3x^2 - 5x - 10 > 2$$
- $$3x^2 - 5x - 8 > 0$$
- $$3x^2 - 5x - 8 = 0$$
- $$(3x - 8)(x + 1) = 0$$
- $$3x - 8 = 0 \quad \text{or} \quad x + 1 = 0$$
- $$3x = 8 \qquad \qquad x = -1$$
- $$x = \frac{8}{3}$$
- Choose test points -2 , 0 , and 3 from the intervals $x < -1$, $-1 < x < \frac{8}{3}$, and $x > \frac{8}{3}$, respectively.
- The values of x less than -1 satisfy the inequality $3x^2 - 5x - 10 > 2$.
- a)** Upon verification, Dolores realized she made an error. Explain the error and provide a correct solution.
- b)** Use a different strategy to determine the solution to $3x^2 - 5x - 10 > 2$.
- 17.** A scoop in field hockey occurs when a player lifts the ball off the ground with a shovel-like movement of the stick, which is placed slightly under the ball. Suppose a player passes the ball with a scoop modelled by the function $h(t) = -4.9t^2 + 10.4t$, where h is the height of the ball, in metres, and t represents time, in seconds. For what length of time, to the nearest hundredth of a second, is the ball above 3 m?