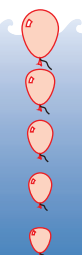# Absolute Value and Reciprocal Functions

Suppose you and your friend each live 2 km from your school, but in opposite directions from the school. You could represent these distances as 2 km in one direction and −2 km in the other direction. However, you would both say that you live the same distance, 2 km, from your school. The *absolute value* of the distance each of you lives from your school is 2 km. Can you think of other examples where you would use an absolute value?

Currency exchange is an example of a reciprocal relationship. If 1 euro is equivalent to 1.3 Canadian dollars, what is 1 Canadian dollar worth in euros? If you take a balloon underwater, you can represent the relationship between its shrinking volume and the increasing pressure of the air inside the balloon as a reciprocal function.

Is this depth change 20 m or −20 m? Which value would you use? Why?

Depth (m)	Pressure (atm)	Air Volume (m³)
0	1	1
10	2	$\frac{1}{2}$
20	3	$\frac{1}{3}$
30	4	$\frac{1}{4}$
40	5	$\frac{1}{5}$

In this chapter, you will learn about absolute value and reciprocal functions. You will also learn how they are used to solve problems.

Web Link

The relationship between the pressure and the volume of a confined gas held at a constant temperature is known as Boyle's law. To learn more about Boyle's law, go to www.mhrprecalc11.ca and follow the links.

Key Terms

absolute value

absolute value function

piecewise function

invariant point

absolute value equation

reciprocal function

asymptote

Absolute Value

Focus on...

- determining the absolute values of numbers and expressions
- explaining how the distance between two points on a number line can be expressed in terms of absolute value
- comparing and ordering the absolute values of real numbers in a given set

The hottest temperature ever recorded in Saskatoon, Saskatchewan, was 40.6 °C on June 5, 1988. The coldest temperature, −50.0 °C, was recorded on February 1, 1893. You can calculate the total temperature difference as

$$-50.0 - 40.6 = d \quad \text{or} \quad 40.6 - (-50.0) = d$$
$$-90.6 = d \qquad\qquad\qquad 90.6 = d$$

Generally, you use the positive value, 90.6 °C, when describing the difference. Why do you think this is the case? Does it matter which value you use when describing this situation? Can you describe a situation where you would use the negative value?

> **Did You Know?**
>
> In 1962 in Pincher Creek, Alberta, a chinook raised the temperature by 41 °C (from −19 °C to +22 °C) in 1 h. This is a Canadian record for temperature change in a day.

Delta Bessborough Hotel, Saskatoon, Saskatchewan

1. Draw a number line on grid paper that is approximately 20 units long. Label the centre of the number line as 0. Label the positive and negative values on either side of zero, as shown.

Materials
- grid paper
- ruler
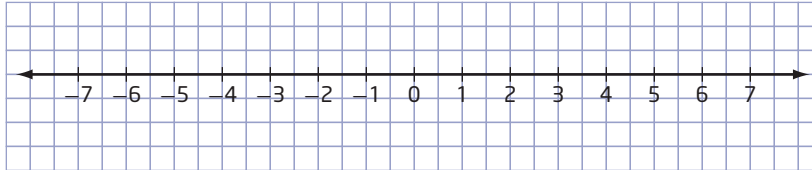
2. Mark the values +4 and −4 on your number line. Describe their distances from 0.

3. a) Plot two points to the right of zero. How many units are between the two points?

   b) Calculate the distance between the two points in two different ways.

4. Repeat step 3 using two points to the left of zero.

5. Repeat step 3 using one point to the right of zero and one point to the left.

6. What do you notice about the numerical values of your calculations and the number of units between each pair of points you chose in steps 3, 4, and 5?

7. What do you notice about the signs of the two calculated distances for each pair of points in steps 3, 4, and 5?

Reflect and Respond

8. Identify three different sets of points that have a distance of 5 units between them. Include one set of points that are both positive, one set of points that are both negative, and one set containing a positive and a negative value. How did you determine each set of points?

9. Explain why the distance from 0 to +3 is the same as the distance from 0 to −3. Why is the distance referred to as a positive number?

absolute value

- $|a| = \begin{cases} a, \text{ if } a \geq 0 \\ -a, \text{ if } a < 0 \end{cases}$

For a real number a, the **absolute value** is written as $|a|$ and is a positive number.

Two vertical bars around a number or expression are used to represent the absolute value of the number or expression.

For example,
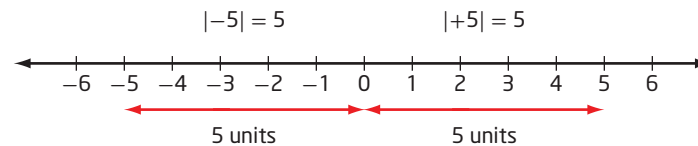- The absolute value of a positive number is the positive number.
  $|+5| = 5$
- The absolute value of zero is zero.
  $|0| = 0$
- The absolute value of a negative number is the negative of that number, resulting in the positive value of that number.
  $|-5| = -(-5)$
  $ = 5$

Absolute value can be used to represent the distance of a number from zero on a real-number line.

$$|-5| = 5 \qquad\qquad |+5| = 5$$

5 units   5 units

In Chapter 5, you learned that $\sqrt{x^2} = x$ only when x is positive.

How can you use this fact and the definition of absolute value to show that $\sqrt{x^2} = |x|$?

In general, the absolute value of a real number a is defined as
$$|a| = \begin{cases} a, \text{ if } a \geq 0 \\ -a, \text{ if } a < 0 \end{cases}$$

Example 1

Determining the Absolute Value of a Number

Evaluate the following.
a) $|3|$
b) $|-7|$

Solution

a) $|3| = 3$ since $|a| = a$ for $a \geq 0$.

b) $|-7| = -(-7)$
$ = 7$
since $|a| = -a$ for $a < 0$.

Your Turn

Evaluate the following.
a) $|9|$
b) $|-12|$

Example 2

Compare and Order Absolute Values

Write the real numbers in order from least to greatest.

$$\left|-6.5\right|, 5, \left|4.75\right|, -3.4, \left|-\frac{12}{5}\right|, \left|-0.1\right|, -0.01, \left|-2\frac{1}{2}\right|$$

Solution

First, evaluate each number and express it in decimal form.
$6.5, 5, 4.75, -3.4, 2.4, 0.1, -0.01, 2.5$

Then, rearrange from least to greatest value.
$-3.4, -0.01, 0.1, 2.4, 2.5, 4.75, 5, 6.5$

Now, show the original numbers in order from least to greatest.
$$-3.4, -0.01, \left|-0.1\right|, \left|-\frac{12}{5}\right|, \left|-2\frac{1}{2}\right|, \left|4.75\right|, 5, \left|-6.5\right|$$

Your Turn

Write the real numbers in order from least to greatest.

$$\left|3.5\right|, -2, \left|-5.75\right|, 1.05, \left|-\frac{13}{4}\right|, \left|-0.5\right|, -1.25, \left|-3\frac{1}{3}\right|$$

Absolute value symbols should be treated in the same manner as brackets. Evaluate the absolute value of a numerical expression by first applying the order of operations inside the absolute value symbol, and then taking the absolute value of the result.

Example 3

Evaluating Absolute Value Expressions

Evaluate the following.

a) $\left|4\right| - \left|-6\right|$ **b)** $5 - 3\left|2 - 7\right|$ **c)** $\left|-2(5 - 7)^2 + 6\right|$

Solution

a) $\left|4\right| - \left|-6\right| = 4 - 6$
$$= -2$$

b) $5 - 3\left|2 - 7\right| = 5 - 3\left|-5\right|$
$$= 5 - 3(5)$$
$$= 5 - 15$$
$$= -10$$

c) $\left|-2(5 - 7)^2 + 6\right| = \left|-2(-2)^2 + 6\right|$
$$= \left|-2(4) + 6\right|$$
$$= \left|-8 + 6\right|$$
$$= \left|-2\right|$$
$$= 2$$

Apply the order of operations to evaluate the expression inside the absolute value symbol.

Your Turn

Evaluate the following.

a) $\left|-4\right| - \left|-3\right|$ **b)** $\left|-12 + 8\right|$ **c)** $\left|12(-3) + 5^2\right|$

Example 4

Change in Stock Value

On stock markets, individual stock and bond values fluctuate a great deal, especially when the markets are volatile. A particular stock on the Toronto Stock Exchange (TSX) opened the month at $13.55 per share, dropped to $12.70, increased to $14.05, and closed the month at $13.85. Determine the total change in the value of this stock for the month. This total shows how active the stock was that month.

Solution

Represent the stock values by $V_1 = 13.55$, $V_2 = 12.70$, $V_3 = 14.05$, and $V_4 = 13.85$. Calculate each change in stock value using $|V_{i+1} - V_i|$, where $i = 1, 2, 3$.

Calculate each change in stock value and find the sum of these changes.

Does the order in which the values are subtracted matter?

$$|V_2 - V_1| + |V_3 - V_2| + |V_4 - V_3|$$
$$= |12.70 - 13.55| + |14.05 - 12.70| + |13.85 - 14.05|$$
$$= |-0.85| + |+1.35| + |-0.20|$$
$$= 0.85 + 1.35 + 0.20$$
$$= 2.40$$

The total change in stock value for the month is $2.40.

Why would an investor find the volatility of a particular stock useful when making investment decisions?

Your Turn

Wesley volunteers at a local hospital because he is interested in a career in health care. One day, he takes the elevator from the first floor up to the sixth floor to see his supervising nurse. His list of tasks for that day sends him down to the second floor to work in the gift shop, up to the fourth floor to visit with patients, and down to the first floor to greet visitors and patients. What is the total change in floors for Wesley that day?

Key Ideas

- The absolute value of a real number a is defined as

$$|a| = \begin{cases} a, \text{ if } a \geq 0 \\ -a, \text{ if } a < 0 \end{cases}$$

- Geometrically, the absolute value of a real number a, written as $|a|$, is its distance from zero on the number line, regardless of direction.

- Determine the absolute value of a numerical expression by
  - evaluating the numerical expression inside the absolute value symbol using the order of operations
  - taking the absolute value of the resulting expression

Check Your Understanding

Practise

1. Evaluate.

a) $|9|$ **b)** $|0|$

c) $|-7|$ **d)** $|-4.728|$

e) $|6.25|$ **f)** $\left|-5\frac{1}{2}\right|$

2. Order the numbers from least to greatest.

$|0.8|, 1.1, |-2|, \left|\frac{3}{5}\right|, -0.4, \left|-1\frac{1}{4}\right|, -0.8$

3. Order the numbers from greatest to least.

$-2.4, |1.3|, \left|-\frac{7}{5}\right|, -1.9, |-0.6|, \left|1\frac{1}{10}\right|, 2.2$

4. Evaluate each expression.

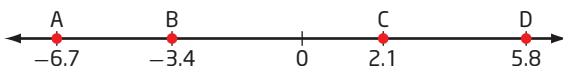**a)** $|8 - 15|$ **b)** $|3| - |-8|$

c) $|7 - (-3)|$ **d)** $|2 - 5(3)|$

5. Use absolute value symbols to write an expression for the distance between each pair of specified points on the number line. Determine the distance.

A   B   C   D

−6.7   −3.4   0   2.1   5.8

a) A and C   **b)** B and D

c) C and B   **d)** D and A

6. Determine the value of each absolute value expression.

a) $2|-6 - (-11)|$

b) $|-9.5| - |12.3|$

c) $3\left|\frac{1}{2}\right| + 5\left|-\frac{3}{4}\right|$

d) $|3(-2)^2 + 5(-2) + 7|$

e) $|-4 + 13| + |6 - (-9)| - |8 - 17| + |-2|$

Apply

7. Use absolute value symbols to write an expression for the length of each horizontal or vertical line segment. Determine each length.

a) A(8, 1) and B(3, 1)

b) A(12, 9) and B(−8, 9)

c) A(6, 2) and B(6, 9)

d) A(−1, −7) and B(−1, 15)

e) A(a, y) and B(b, y)

f) A(x, m) and B(x, n)

8. Southern Alberta often experiences dry chinook winds in winter and spring that can change temperatures by a large amount in a short time. On a particular day in Warner, Alberta, the temperature was −11 °C in the morning. A chinook wind raised the temperature to +7 °C by afternoon. The temperature dropped to −9 °C during the night. Use absolute value symbols to write an expression for the total change in temperature that day. What is the total change in temperature for the day?

Did You Know?

A First Nations legend of the St'at'imc Nation of British Columbia says that a girl named Chinook-Wind married Glacier and moved to his country. In this foreign land, she longed for her home and sent a message to her people. They came to her first in a vision of snowflakes, then rain, and finally as a melting glacier that took her home.

9. Suppose a straight stretch of highway running west to east begins at the town of Allenby (0 km). The diagram shows the distances from Allenby east to various towns. A new grain storage facility is to be built along the highway 24 km east of Allenby. Write an expression using absolute value symbols to determine the total distance of the grain storage facility from all seven towns on the highway for this proposed location. What is the distance?

0	10	17	30	42	55	72
Allenby	Birkend	Crawley	Denford	Essex	Fortier	Grey Ridge
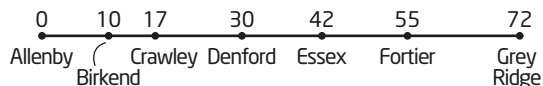
10. The Alaska Highway runs from Dawson Creek, British Columbia, to Delta Junction, Alaska. Travel guides along the highway mark historic mileposts, from mile 0 in Dawson Creek to mile 1422 in Delta Junction. The table shows the Ramsay family's trip along this highway.

	Destination	Mile Number
Starting Point	Charlie Lake campground	51
Tuesday	Liard River, British Columbia	496
Wednesday	Whitehorse, Yukon Territory	918
Thursday	Beaver Creek, Yukon Territory	1202
	Haines Junction, Yukon Territory	1016
Friday	Delta Junction, Alaska	1422

Use an expression involving absolute value symbols to determine the total distance, in miles, that the Ramsay family travelled in these four days.

Did You Know?

In 1978, the mileposts along the Canadian section were replaced with kilometre posts. Some mileposts at locations of historic significance remain, although reconstruction and rerouting mean that these markers no longer represent accurate driving distances.

11. When Vanessa checks her bank account on-line, it shows the following balances:

Date	Balance
Oct. 4	$359.22
Oct. 12	$310.45
Oct. 17	$295.78
Oct. 30	$513.65
Nov. 5	$425.59

a) Use an absolute value expression to determine the total change in Vanessa's bank balance during this period.

b) How is this different from the net change in her bank balance?

12. In physics, the amplitude of a wave is measured as the absolute value of the difference between the crest height and the trough height of the wave, divided by 2.

$$\text{Amplitude} = \frac{|\text{crest height} - \text{trough height}|}{2}$$
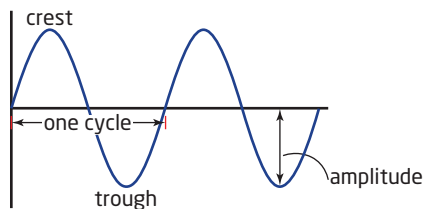
Determine the amplitude of waves with the following characteristics.

a) crest at height 17 and trough at height 2

b) crest at height 90 and trough at height −90

c) crest at height 1.25 and trough at height −0.5

13. The Festival du Voyageur is an annual Francophone winter festival held in Manitoba. One of the outdoor events at the festival is a snowshoe race. A possible trail for the race is 2 km long, with the start at 0 km; checkpoints at 500 m, 900 m, and 1600 m; and the finish at 2 km. Suppose a race organizer travels by snowmobile from the start to the 1600-m checkpoint, back to the 900-m checkpoint, then out to the finish line, and finally back to the 500-m checkpoint. Use an absolute value expression to determine the total distance travelled by the race organizer, in metres and in kilometres.

14. The Yukon Quest dog sled race runs between Fairbanks, Alaska, and Whitehorse, Yukon Territory, a distance of more than 1000 mi. It lasts for 2 weeks. The elevation at Fairbanks is 440 ft, and the elevation at Whitehorse is 2089 ft.
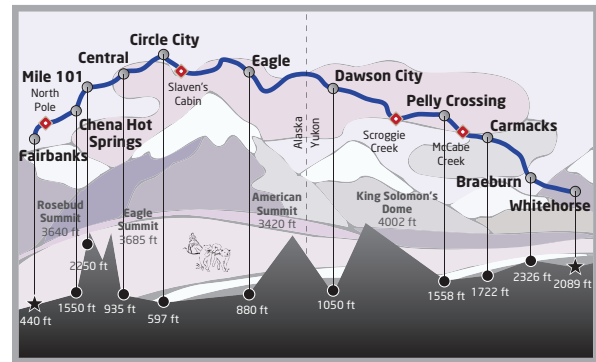
a) Determine the net change in elevation from Fairbanks to Whitehorse.

b) The race passes through Central, at an elevation of 935 ft; Circle City, whose elevation is 597 ft; and Dawson City, at an elevation of 1050 ft. What is the total change in elevation from Fairbanks to Whitehorse, passing through these cities?

Did You Know?

The Yukon Quest has run every year since 1984. The race follows the historic Gold Rush and mail delivery dog sled routes from the turn of the 20th century. In even-numbered years, the race starts in Fairbanks and ends in Whitehorse. In odd-numbered years, the race starts in Whitehorse and ends in Fairbanks.

Extend

15. A trading stock opens the day at a value of $7.65 per share, drops to $7.28 by noon, then rises to $8.10, and finally falls an unknown amount to close the trading day. If the absolute value of the total change for this stock is $1.55, determine the amount that the stock dropped in the afternoon before closing.

16. As part of a scavenger hunt, Toby collects items along a specified trail. Starting at the 2-km marker, he bicycles east to the 7-km marker, and then turns around and bicycles west back to the 3-km marker. Finally, Toby turns back east and bicycles until the total distance he has travelled is 15 km.

   a) How many kilometres does Toby travel in the last interval?

   b) At what kilometre marker is Toby at the end of the scavenger hunt?

17. Mikhala and Jocelyn are examining some effects of absolute value on the sum of the squares of the set of values {−2.5, 3, −5, 7.1}. Mikhala takes the absolute value of each number and then squares it. Jocelyn squares each value and then takes the absolute value.

   a) What result does each student get?

   b) Explain the results.

   c) Is this always true? Explain.

18. a) Michel writes the expression $|x - 5|$, where x is a real number, without the absolute value symbol. His answer is shown below.

$$|x - 5| = \begin{cases} x - 5, \text{ if } x - 5 \geq 0 \\ -(x - 5), \text{ if } x - 5 < 0 \end{cases}$$

$$|x - 5| = \begin{cases} x - 5, \text{ if } x \geq 5 \\ 5 - x, \text{ if } x < 5 \end{cases}$$

Explain the steps in Michel's solution.

   b) If x is a real number, then write each of the following without the absolute value symbol.

   i) $|x - 7|$ **ii)** $|2x - 1|$

   iii) $|3 - x|$ **iv)** $|x^2 + 4|$

19. Julia states, "To determine the absolute value of any number, change the sign of the number." Use an example to show that Julia is incorrect. In your own words, correctly complete the statement, "To determine the absolute value of a number, …."

20. In the Millikan oil drop experiment, oil drops are electrified with either a positive or a negative charge and then sprayed between two oppositely charged metal plates. Since an individual oil drop will be attracted by one plate and repelled by the other plate, the drop will move either upward or downward toward the plate of opposite charge. If the charge on the plates is reversed, the oil drop is forced to reverse its direction and thus stay suspended indefinitely.
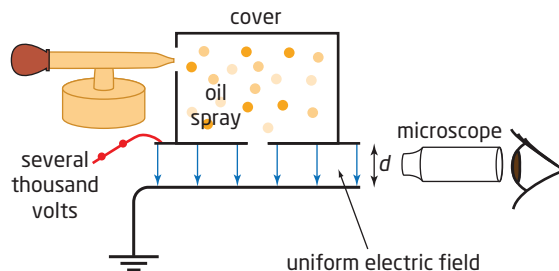
Suppose an oil drop starts out at 45 mm below the upper plate, moves to a point 67 mm below the upper plate, then to a point 32 mm from the upper plate, and finally to a point 58 mm from the upper plate. What total distance does the oil drop travel during the experiment?

Create Connections

21. Describe a situation in which the absolute value of a measurement is preferable to the actual signed value.

22. When an object is thrown into the air, it moves upward, and then changes direction as it returns to Earth. Would it be more appropriate to use signed values (+ or −) or the absolute value for the velocity of the object at any point in its flight? Why? What does the velocity of the object at the top of its flight have to be?

23. A school volleyball team has nine players. The heights of the players are 172 cm, 181 cm, 178 cm, 175 cm, 180 cm, 168 cm, 177 cm, 175 cm, and 178 cm.

a) What is the mean height of the players?

b) Determine the absolute value of the difference between each individual's height and the mean. Determine the sum of the values.

c) Divide the sum by the number of players.

d) Interpret the result in part c) in terms of the height of players on this team.

24. When writing a quadratic function in vertex form, $y = a(x - p)^2 + q$, the vertex of the graph is located at (p, q). If the function has zeros, or its graph has x-intercepts, you can find them using the equation $x = p \pm \sqrt{\left|\frac{q}{a}\right|}$.

a) Use this equation to find the zeros of each quadratic function.

  i) $y = 2(x + 1)^2 - 8$

  ii) $y = -(x + 2)^2 + 9$

  How could you verify that the zeros are correct?

b) What are the zeros of the function $y = 4(x - 3)^2 + 16$? Explain whether or not you could use this method to determine the zeros for all quadratic functions written in vertex form.

25. Explain, using examples, why $\sqrt{x^2} = |x|$.

Project Corner   Space Tourism

*Assume the following for the future of space tourism.*

- The comfort and quality of a cruise ship are available for travel in outer space.

- The space vehicle in which tourists travel is built within specified tolerances for aerodynamics, weight, payload capacity, and life-support systems, to name a few criteria.

- Each space vehicle leaves Earth within a given launch window.

- The distance of Earth from other celestial destinations changes at different times of the year.

- The quantity of fuel on board the space vehicle determines its range of travel.

- How might absolute value be involved in these design and preparation issues?

Absolute Value Functions

7.2

Focus on...

- creating a table of values for $y = |f(x)|$, given a table of values for $y = f(x)$
- sketching the graph of $y = |f(x)|$ and determining its intercept(s), domain, and range
- generalizing a rule for writing absolute value functions in piecewise notation
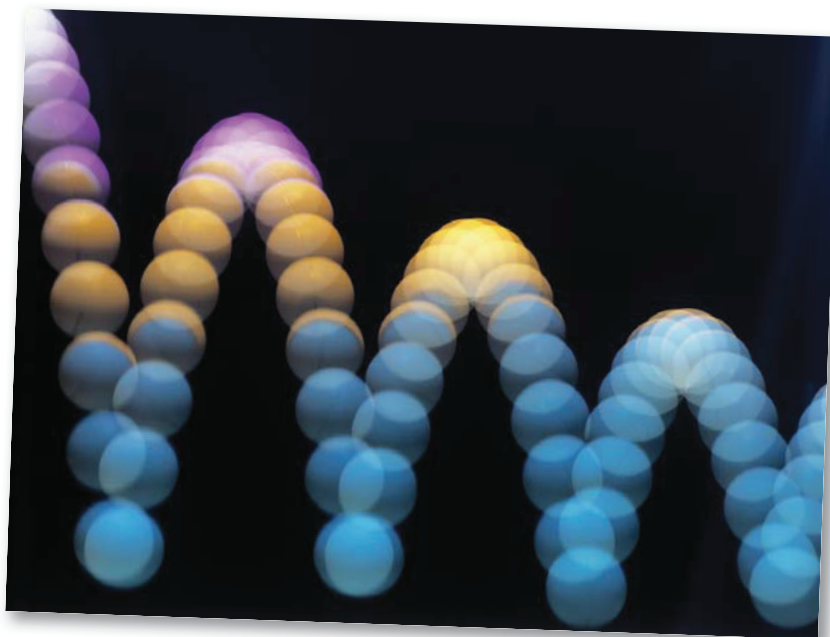
Stroboscopic photography involves using a flashing strobe light and a camera with an open shutter. You must take stroboscopic photographs in darkness so that every time the strobe flashes, you take a still image of a moving object at that instant. Shown here is a stroboscopic photograph following the path of a bouncing ball. To measure the total vertical distance the ball travels as it bounces over a certain time interval, use the absolute value of the function that models the height over time. What type of function would you use to model the height of this bouncing ball over time?

Investigate Absolute Value Functions

Materials

- grid paper

In this activity, you will explore the similarities and differences between linear, quadratic, and absolute value functions.

Part A: Compare Linear Functions With Corresponding Absolute Value Functions

Consider the functions $f(x) = x$ and $g(x) = |x|$.

1. Copy the table of values. Use the values of $f(x)$ to determine the values of $g(x)$ and complete the table.

What happens to the value of the functions $f(x)$ and $g(x)$ when the values of x are negative?

x	f(x)	g(x)
−3	−3	
−2	−2	
−1	−1	
0	0	
1	1	
2	2	
3	3	

2. Use the coordinate pairs to sketch graphs of the functions on the same grid.

3. Which characteristics of the two graphs are similar and which are different?

4. From the graph, explain why the absolute value relation is a function.

5. **a)** Describe the shape of the graph of $g(x)$.

   b) If you could sketch the graph of $g(x)$ using two linear functions, what would they be? Are there any restrictions on the domain and range of each function? If so, what are they?

Part B: Compare Quadratic Functions With Corresponding Absolute Value Functions

Consider the functions $f(x) = x^2 - 3$ and $h(x) = |x^2 - 3|$.

6. Copy the table of values. Use the values of $f(x)$ to determine the values of $h(x)$ and complete the table.

x	f(x)	h(x)
−3	6	
−2	1	
−1	−2	
0	−3	
1	−2	
2	1	
3	6	

When are the values of $f(x)$ and $h(x)$ the same and when are they different?

7. Use the coordinate pairs to sketch the graphs of $f(x)$ and $h(x)$ on the same grid.

8. Which characteristics of the two graphs are similar and which are different?

9. **a)** For what values of x are the graphs of $f(x)$ and $h(x)$ the same? different?

   b) If you could sketch the graph of $h(x)$ using two quadratic functions, what would they be? Are there any restrictions on the domain and range of each function? If so, what are they?

10. Describe how the graph of a linear or quadratic function is related to its corresponding absolute value graph.

absolute value function

- a function that involves the absolute value of a variable

The vertex, (0, 0), divides the graph of this **absolute value function** $y = |x|$ into two distinct pieces.
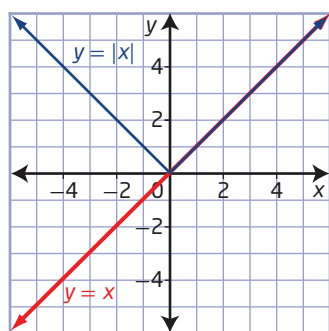
For all values of x less than zero, the y-value is $-x$. For all values of x greater than or equal to zero, the y-value is x. Since the function is defined by two different rules for each interval in the domain, you can define $y = |x|$ as the **piecewise function**

piecewise function

- a function composed of two or more separate functions or *pieces*, each with its own specific domain, that combine to define the overall function

- the absolute value function $y = |x|$ can be defined as the piecewise function
$y = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$

$$y = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$$

The graph shows how $y = |x|$ is related to the graph of $y = x$. Since $|x|$ cannot be negative, the part of the graph of $y = x$ that is below the x-axis is reflected in the x-axis to become the line $y = -x$ in the interval $x < 0$. The part of the graph of $y = x$ that is on or above the x-axis is zero or positive and remains unchanged as the line $y = x$ in the interval $x \geq 0$.

Example 1

Graph an Absolute Value Function of the Form $y = |ax + b|$

Consider the absolute value function $y = |2x - 3|$.
a) Determine the y-intercept and the x-intercept.
b) Sketch the graph.
c) State the domain and range.
d) Express as a piecewise function.

Solution

a) To determine the y-intercept, let $x = 0$ and solve for y.
$y = |2x - 3|$
$y = |2(0) - 3|$
$y = |-3|$
$y = 3$
The y-intercept occurs at (0, 3).

To determine the x-intercept, set $y = 0$ and solve for x.

$|2x - 3| = 0$

$2x - 3 = 0$ Since $|0| = 0$, $|2x - 3| = 0$ when $2x - 3 = 0$.

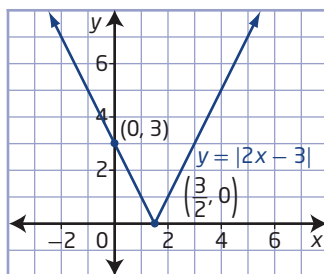$2x = 3$

$x = \dfrac{3}{2}$

The x-intercept occurs at $\left(\dfrac{3}{2}, 0\right)$.

b) **Method 1: Sketch Using a Table of Values**

Create a table of values, using the x-intercept and values to the right and left of it.

Sketch the graph using the points in the table.

x	$y = |2x - 3|$
-1	5
0	3
$\dfrac{3}{2}$	0
3	3
4	5

Method 2: Sketch Using the Graph of $y = 2x - 3$

Use the graph of $y = 2x - 3$ to graph $y = |2x - 3|$.
Sketch the graph of $y = 2x - 3$, which is a line with a slope of 2 and a y-intercept of -3.

The x-intercept of the original function is the x-intercept of the corresponding absolute value function. The point representing the x-intercept is an **invariant point**.

Reflect in the x-axis the part of the graph of $y = 2x - 3$ that is below the x-axis.

invariant point

- a point that remains unchanged when a transformation is applied to it
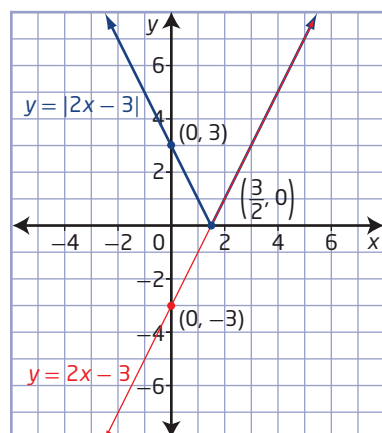
What other points on the graph are invariant points?

c) Since there is no *x*-value that cannot be substituted into the function $y = |2x - 3|$, the domain is all real numbers, or $\{x \mid x \in R\}$. For all values of *x*, $|2x - 3| \geq 0$. The range is $\{y \mid y \geq 0, y \in R\}$.

d) The V-shaped graph of the absolute value function $y = |2x - 3|$ is composed of two separate linear functions, each with its own domain.

- When $x \geq \dfrac{3}{2}$, the graph of $y = |2x - 3|$ is the graph of $y = 2x - 3$, which is a line with a slope of 2 and a *y*-intercept of −3.

- When $x < \dfrac{3}{2}$, the graph of $y = |2x - 3|$ is the graph of $y = 2x - 3$ reflected in the *x*-axis. The equation of the reflected graph is $y = -(2x - 3)$ or $y = -2x + 3$, which is a line with a slope of −2 and a *y*-intercept of 3.

You can combine these two linear functions with their domains to define the absolute value function $y = |2x - 3|$. Express the absolute value function $y = |2x - 3|$ as the piecewise function

$$y = \begin{cases} 2x - 3, & \text{if } x \geq \dfrac{3}{2} \\ -(2x - 3), & \text{if } x < \dfrac{3}{2} \end{cases}$$

Your Turn

Consider the absolute value function $y = |3x + 1|$.
a) Determine the *y*-intercept and the *x*-intercept.
b) Sketch the graph.
c) State the domain and range.
d) Express as a piecewise function.

Example 2

Graph an Absolute Value Function of the Form $f(x) = |ax^2 + bx + c|$

Consider the absolute value function $f(x) = |-x^2 + 2x + 8|$.
a) Determine the *y*-intercept and the *x*-intercepts.
b) Sketch the graph.
c) State the domain and range.
d) Express as a piecewise function.

Solution

a) Determine the *y*-intercept by evaluating the function at $x = 0$.
$f(x) = |-x^2 + 2x + 8|$
$f(0) = |-(0)^2 + 2(0) + 8|$
$f(0) = |8|$
$f(0) = 8$
The *y*-intercept occurs at $(0, 8)$

The x-intercepts are the real zeros of the function, since they correspond to the x-intercepts of the graph.

$$f(x) = |-x^2 + 2x + 8|$$
$$0 = -x^2 + 2x + 8$$
$$0 = -(x^2 - 2x - 8)$$
$$0 = -(x + 2)(x - 4)$$
$$x + 2 = 0 \quad \text{or} \quad x - 4 = 0$$
$$x = -2 \qquad\qquad x = 4$$

The x-intercepts occur at $(-2, 0)$ and $(4, 0)$.

b) Use the graph of $y = f(x)$ to graph $y = |f(x)|$.

Complete the square to convert the quadratic function $y = -x^2 + 2x + 8$ to vertex form, $y = a(x - p)^2 + q$.

$$y = -x^2 + 2x + 8$$
$$y = -(x^2 - 2x) + 8$$
$$y = -(x^2 - 2x + 1 - 1) + 8$$
$$y = -[(x^2 - 2x + 1) - 1] + 8$$
$$y = -[(x - 1)^2 - 1] + 8$$
$$y = -(x - 1)^2 - 1(-1) + 8$$
$$y = -(x - 1)^2 + 9$$

Since $p = 1$ and $q = 9$, the vertex is located at $(1, 9)$. Since $a < 0$, the parabola opens downward. Sketch the graph.
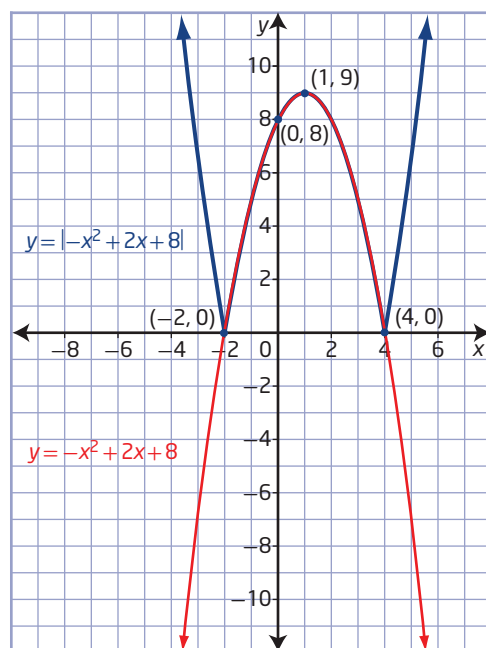
What other methods could you use to find the vertex of the quadratic function?

Reflect in the x-axis the part of the graph of $y = -x^2 + 2x + 8$ that lies below the x-axis.

c) The domain is all real numbers, or $\{x \mid x \in \mathrm{R}\}$, and the range is all non-negative values of y, or $\{y \mid y \geq 0, y \in \mathrm{R}\}$.

d) The graph of $y = |-x^2 + 2x + 8|$ consists of two separate quadratic functions. You can use the x-intercepts to identity each function's specific domain.

- When $-2 \leq x \leq 4$, the graph of $y = |-x^2 + 2x + 8|$ is the graph of $y = -x^2 + 2x + 8$, which is a parabola opening downward with a vertex at $(1, 9)$, a y-intercept of 8, and x-intercepts at -2 and 4.

- When $x < -2$ or $x > 4$, the graph of $y = |-x^2 + 2x + 8|$ is the graph of $y = -x^2 + 2x + 8$ reflected in the x-axis. The equation of the reflected graph is $y = -(-x^2 + 2x + 8)$ or $y = x^2 - 2x - 8$, which is a parabola opening upward with a vertex at $(1, -9)$, a y-intercept of -8, and x-intercepts at -2 and 4.
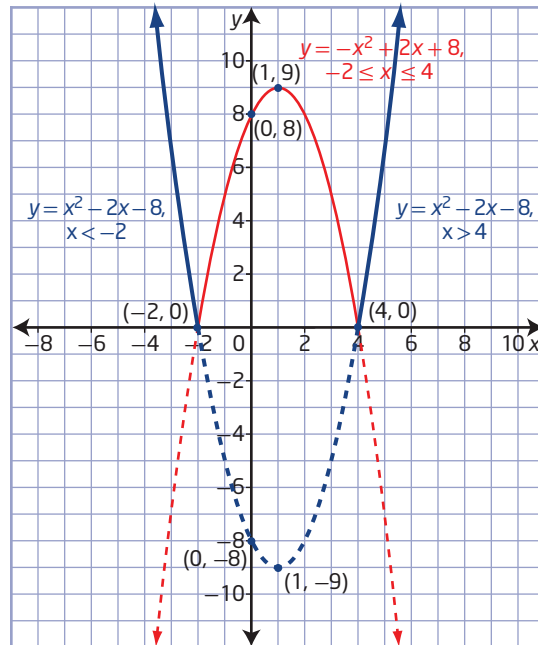
Express the absolute value function $y = |-x^2 + 2x + 8|$ as the piecewise function

$$y = \begin{cases} -x^2 + 2x + 8, & \text{if } -2 \leq x \leq 4 \\ -(-x^2 + 2x + 8), & \text{if } x < -2 \text{ or } x > 4 \end{cases}$$

Your Turn

Consider the absolute value function $f(x) = |x^2 - x - 2|$.
a) Determine the y-intercept and the x-intercepts.
b) Sketch the graph.
c) State the domain and range.
d) Express as a piecewise function.

- You can analyse absolute value functions in several ways:
  - graphically, by sketching and identifying the characteristics of the graph, including the x-intercepts and the y-intercept, the minimum values, the domain, and the range
  - algebraically, by rewriting the function as a piecewise function
  - In general, you can express the absolute value function $y = |f(x)|$ as the piecewise function

  $$y = \begin{cases} f(x), \text{ if } f(x) \geq 0 \\ -f(x), \text{ if } f(x) < 0 \end{cases}$$

- The domain of an absolute value function $y = |f(x)|$ is the same as the domain of the function $y = f(x)$.

- The range of an absolute value function $y = |f(x)|$ depends on the range of the function $y = f(x)$. For the absolute value of a linear or quadratic function, the range will generally, but not always, be $\{y \mid y \geq 0, y \in R\}$.

Check Your Understanding

Practise

1. Given the table of values for $y = f(x)$, create a table of values for $y = |f(x)|$.

a)

x	y = f(x)
−2	−3
−1	−1
0	1
1	3
2	5

b)

x	y = f(x)
−2	0
−1	−2
0	−2
1	0
2	4

2. The point $(-5, -8)$ is on the graph of $y = f(x)$. Identify the corresponding point on the graph of $y = |f(x)|$.

3. The graph of $y = f(x)$ has an x-intercept of 3 and a y-intercept of -4. What are the x-intercept and the y-intercept of the graph of $y = |f(x)|$?

4. The graph of $y = f(x)$ has x-intercepts of -2 and 7, and a y-intercept of $-\dfrac{3}{2}$. State the x-intercepts and the y-intercept of the graph of $y = |f(x)|$.
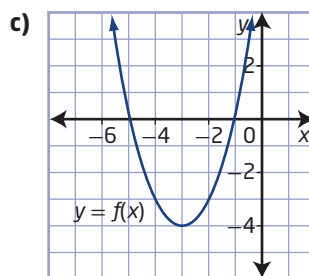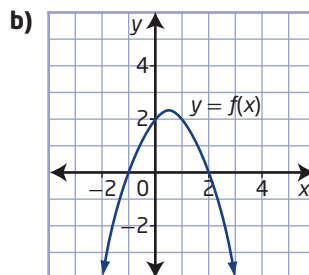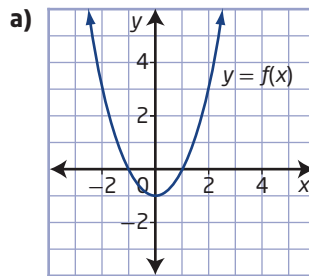
5. Copy the graph of $y = f(x)$. On the same set of axes, sketch the graph of $y = |f(x)|$.

a)

b)

c)
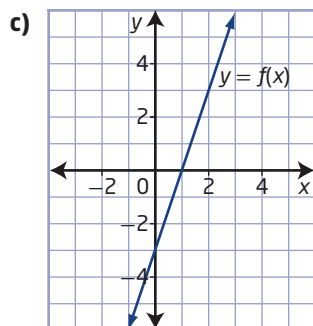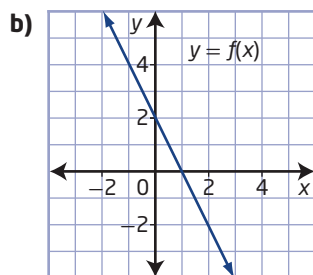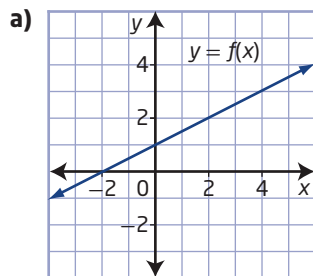
6. Sketch the graph of each absolute value function. State the intercepts and the domain and range.

a) $y = |2x - 6|$

b) $y = |x + 5|$

c) $f(x) = |-3x - 6|$

d) $g(x) = |-x - 3|$

e) $y = \left|\dfrac{1}{2}x - 2\right|$

f) $h(x) = \left|\dfrac{1}{3}x + 3\right|$

7. Copy the graph of $y = f(x)$. On the same set of axes, sketch the graph of $y = |f(x)|$.

a)

b)

c)

8. Sketch the graph of each function. State the intercepts and the domain and range.

a) $y = |x^2 - 4|$
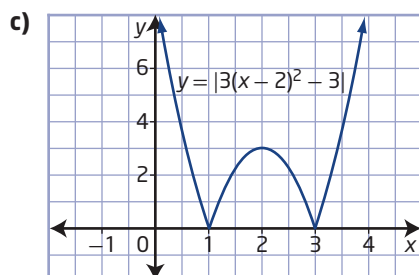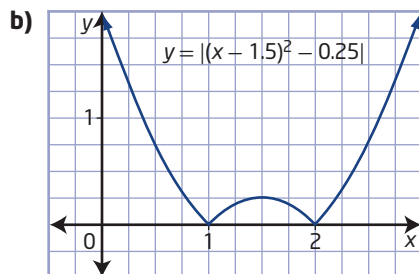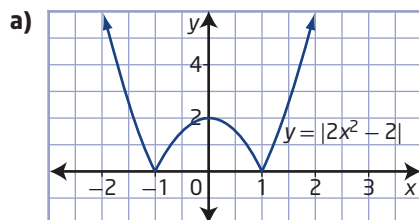
b) $y = |x^2 + 5x + 6|$

c) $f(x) = |-2x^2 - 3x + 2|$

d) $y = \left|\dfrac{1}{4}x^2 - 9\right|$

e) $g(x) = |(x - 3)^2 + 1|$

f) $h(x) = |-3(x + 2)^2 - 4|$

9. Write the piecewise function that represents each graph.

a)

b)

c)
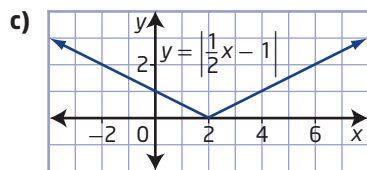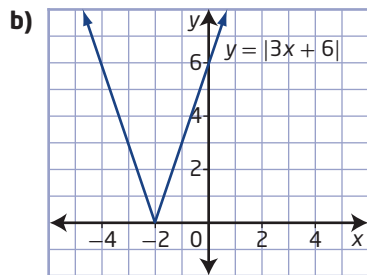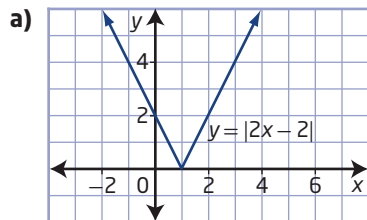
10. What piecewise function could you use to represent each graph of an absolute value function?

a)

b)

c)

11. Express each function as a piecewise function.

a) $y = |x - 4|$

b) $y = |3x + 5|$

c) $y = |-x^2 + 1|$

d) $y = |x^2 - x - 6|$

Apply

12. Consider the function $g(x) = |6 - 2x|$.

a) Create a table of values for the function using values of -1, 0, 2, 3, and 5 for x.

b) Sketch the graph.

c) Determine the domain and range for $g(x)$.

d) Write the function in piecewise notation.

**13.** Consider the function $g(x) = |x^2 - 2x - 8|$.

a) What are the y-intercept and x-intercepts of the graph of the function?

b) Graph the function.

c) What are the domain and range of $g(x)$?

d) Express the function as a piecewise function.

14. Consider the function $g(x) = |3x^2 - 4x - 4|$.

a) What are the intercepts of the graph?

b) Graph the function.

c) What are the domain and range of $g(x)$?

d) What is the piecewise notation form of the function?

15. Raza and Michael are discussing the functions $p(x) = 2x^2 - 9x + 10$ and $q(x) = |2x^2 - 9x + 10|$. Raza says that the two functions have identical graphs. Michael says that the absolute value changes the graph so that $q(x)$ has a different range and a different graph from $p(x)$. Who is correct? Explain your answer.

16. Air hockey is a table game where two players try to score points by hitting a puck into the other player's goal. The diameter of the puck is 8.26 cm. Suppose a Cartesian plane is superimposed over the playing surface of an air hockey table so that opposite corners have the coordinates (0, 114) and (236, 0), as shown. The path of the puck hit by a player is given by $y = |0.475x - 55.1|$.
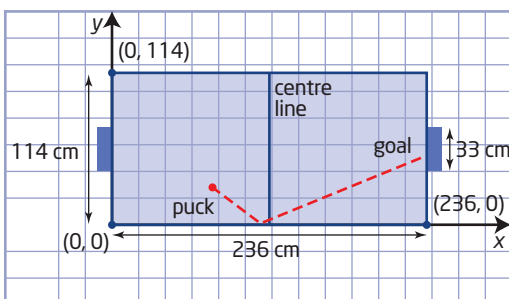
a) Graph the function.

b) At what point does the puck ricochet off the side of the table?

c) If the other player does not touch the puck, verify whether or not the puck goes into the goal.

17. The velocity, v, in metres per second, of a go-cart at a given time, t, in seconds, is modelled by the function $v(t) = -2t + 4$. The distance travelled, in metres, can be determined by calculating the area between the graph of $v(t) = |-2t + 4|$ and the x-axis. What is the distance travelled in the first 5 s?

18. a) Graph $f(x) = |3x - 2|$ and $g(x) = |-3x + 2|$. What do you notice about the two graphs? Explain why.

b) Graph $f(x) = |4x + 3|$. Write a different absolute value function of the form $g(x) = |ax + b|$ that has the same graph.

19. Graph $f(x) = |x^2 - 6x + 5|$. Write a different absolute value function of the form $g(x) = |ax^2 + bx + c|$ that has the same graph as $f(x) = |x^2 - 6x + 5|$.

20. An absolute value function has the form $f(x) = |ax + b|$, where $a \neq 0$, $b \neq 0$, and $a, b \in$ R. If the function $f(x)$ has a domain of $\{x \mid x \in$ R$\}$, a range of $\{y \mid y \geq 0, y \in$ R$\}$, an x-intercept occurring at $\left(\dfrac{3}{2}, 0\right)$, and a y-intercept occurring at (0, 6), what are the values of a and b?

21. An absolute value function has the form $f(x) = |x^2 + bx + c|$, where $b \neq 0$, $c \neq 0$, and $b, c \in$ R. If the function $f(x)$ has a domain of $\{x \mid x \in$ R$\}$, a range of $\{y \mid y \geq 0, y \in$ R$\}$, x-intercepts occurring at (−6, 0) and (2, 0), and a y-intercept occurring at (0, 12), determine the values of b and c.

22. Explain why the graphs of $y = |x^2|$ and $y = x^2$ are identical.

Extend

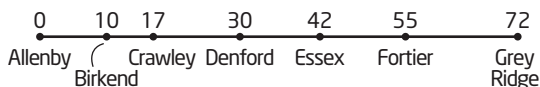**23.** Is the following statement true for all $x, y \in R$? Justify your answer.

$$|x| + |y| = |x + y|$$

24. Draw the graph of $|x| + |y| = 5$.

25. Use the piecewise definition of $y = |x|$ to prove that for all $x, y \in R$, $|x|(|y|) = |xy|$.

26. Compare the graphs of $f(x) = |3x - 6|$ and $g(x) = |3x| - 6$. Discuss the similarities and differences.

Create Connections

27. Explain how to use a piecewise function to graph an absolute value function.

28. Consider the quadratic function $y = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. Describe the nature of the discriminant, $b^2 - 4ac$, for the graphs of $y = ax^2 + bx + c$ and $y = |ax^2 + bx + c|$ to be identical.

29. **MINI LAB** In Section 7.1, you solved the following problem:

Suppose a straight stretch of highway running west to east begins at the town of Allenby (0 km). The diagram shows the distances from Allenby east along the highway to various towns. A new grain storage facility is to be built along the highway 24 km from Allenby. Find the total distance of the grain storage facility from all of the seven towns on the highway for this proposed location.

```
0    10  17     30     42     55         72
●━━━┳━┳━━━●━━━━●━━━━●━━━━●━━━━━━●
Allenby    Crawley Denford Essex  Fortier    Grey
     Birkend                                  Ridge
```

Step 1 Rather than building the facility at a point 24 km east of Allenby, as was originally planned, there may be a more suitable location along the highway that would minimize the total distance of the grain storage facility from all of the towns. Do you think that point exists? If so, predict its location.

Step 2 Let the location of the grain storage facility be at point x (x kilometres east of Allenby). Then, the absolute value of the distance of the facility from Allenby is $|x|$ and from Birkend is $|x - 10|$. Why do you need to use absolute value?

Continue this process to write absolute value expressions for the distance of the storage facility from each of the seven towns. Then, combine them to create a function for the total of the absolute value distances from the different towns to point x.

Step 3 Graph the combined function using a graphing calculator. Set an appropriate window to view the graph. What are the window settings?

Step 4 a) What does the graph indicate about placing the point x at different locations along the highway?

b) What are the coordinates of the minimum point on the graph?

c) Interpret this point with respect to the location of the grain storage facility.

30. Each set of transformations is applied to the graph of $f(x) = x^2$ in the order listed. Write the function of each transformed graph.

a) a horizontal translation of 3 units to the right, a vertical translation of 7 units up, and then take its absolute value

b) a change in the width by a factor of $\frac{4}{5}$, a horizontal translation of 3 units to the left, and then take its absolute value

c) a reflection in the x-axis, a vertical translation of 6 units down, and then take its absolute value

d) a change in the width by a factor of 5, a horizontal translation of 3 units to the left, a vertical translation of 3 units up, and then take its absolute value

Absolute Value Equations

Focus on...

- solving an absolute value equation graphically, with or without technology
- algebraically solving an equation with a single absolute value and verifying the solution
- explaining why the absolute value equation $|f(x)| = b$ for $b < 0$ has no solution

Is the speed of light the maximum velocity possible? According to Albert Einstein's theory of relativity, an object travelling near the speed of light, approximately 300 000 km/s, will move more slowly and shorten in length from the point of view of an observer on Earth. On the television show *Star Trek*, the speed of light was called Warp 1 and the spaceship *USS Enterprise* was able to travel at much greater speeds. Is this possible or just a fantasy?
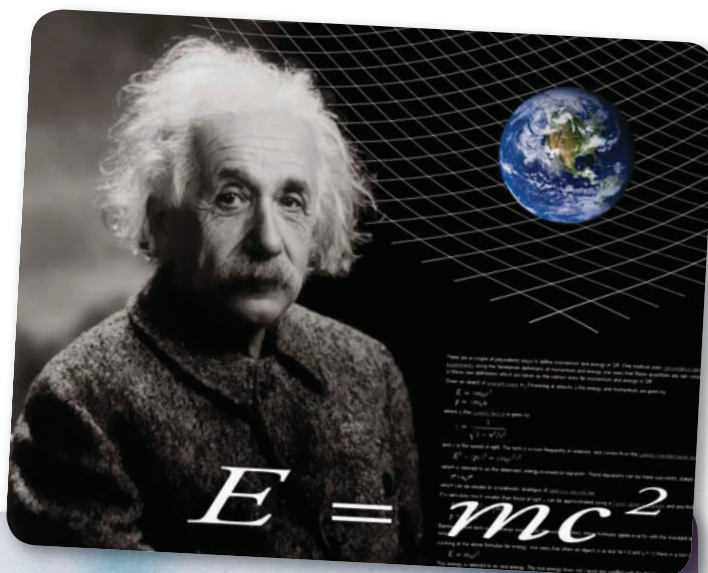
Did You Know?

The town of Vulcan, Alberta, has been using the *Star Trek* connection since the debut of the television series and now receives more than 12 000 visitors per year. There is a replica of the *USS Enterprise* in Vulcan, and the tourism centre is designed as a landing craft. Every year, in June, Vulcan hosts Galaxyfest-Spock Days.

1. Consider the absolute value equation $|x| = 10$.

2. Use the number line to geometrically solve the equation. How many solutions are there?
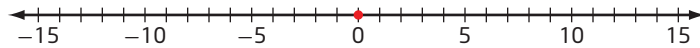
3. How many solutions are there for the equation $|x| = 15$? for $|x| = 5$? for $|x| = b$, $b \neq 0$? What are the solutions?

4. Make a conjecture about the number of solutions for an absolute value equation.

5. Solve the absolute value equation $|x| = 0$.

Reflect and Respond

6. Is it possible to have an absolute value equation that has no solutions? Under what conditions would this happen?

7. Discuss how to use the following definition of absolute value to solve absolute value equations.

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

8. **a)** From the definition of absolute value in step 7, give a general rule for solving $|A| = b$, $b \geq 0$, for A, where A is an algebraic expression.

   b) State a general rule for solving the equation $|A| = b$, $b < 0$, for A.

Link the Ideas

Use the definition of absolute value when solving **absolute value equations** algebraically.

There are two cases to consider.

Case 1: The expression inside the absolute value symbol is positive or zero.

Case 2: The expression inside the absolute value symbol is negative.

absolute value equation

• an equation that includes the absolute value of an expression involving a variable

Example 1

Solve an Absolute Value Equation

Solve $|x - 3| = 7$.

Solution

Method 1: Use Algebra

Using the definition of absolute value,

$$|x - 3| = \begin{cases} x - 3, \text{ if } x \geq 3 \\ -(x - 3), \text{ if } x < 3 \end{cases}$$

Case 1

The expression $|x - 3|$ equals $x - 3$ when $x - 3 \geq 0$, or when $x \geq 3$.

$x - 3 = 7$

$\quad x = 10$

The value 10 satisfies the condition $x \geq 3$.

Case 2

The expression $|x - 3|$ equals $-(x - 3)$ when $x - 3 < 0$, or when $x < 3$.

$-(x - 3) = 7$

$\quad x - 3 = -7$

$\qquad x = -4$

The value -4 satisfies the condition $x < 3$.

Verify the solutions algebraically by substitution.

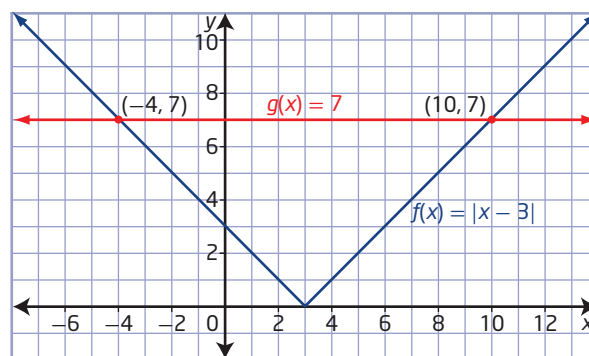| For $x = 10$:		For $x = -4$:	
Left Side	Right Side	Left Side	Right Side
$\quad\lvert x - 3\rvert$	7	$\quad\lvert x - 3\rvert$	7
$= \lvert 10 - 3\rvert$		$= \lvert -4 - 3\rvert$	
$= \lvert 7\rvert$		$= \lvert -7\rvert$	
$= 7$		$= 7$	
Left Side = Right Side		Left Side = Right Side	

The solution is $x = 10$ or $x = -4$.

Method 2: Use a Graph

Graph the functions $f(x) = |x - 3|$ and $g(x) = 7$ on the same coordinate grid to see where they intersect.

Why were these two functions chosen for $f(x)$ and $g(x)$?

The graphs intersect at $(-4, 7)$ and $(10, 7)$. This means that $x = -4$ and $x = 10$ are solutions to the equation $|x - 3| = 7$.

You can verify the solutions using technology. Input the function $f(x) = |x - 3|$ and display the table of values to confirm the solutions you found graphically.

From the table of values, the solution is $x = -4$ or $x = 10$.

x	f1(x):= ▼
	abs(x−3)
−5.	8.
−4.	7.
−3.	6.
−2.	5.
−1.	4.

x	f1(x):= ▼
	abs(x−3)
7.	4.
8.	5.
9.	6.
10.	7.
11.	8.

Your Turn

Solve $|6 - x| = 2$ graphically and algebraically.

Example 2

Solve an Absolute Value Problem

A computerized process controls the amount of batter used to produce cookies in a factory. If the computer program sets the ideal mass before baking at 55 g but allows a tolerance of ± 2.5 g, solve an absolute value equation for the maximum and minimum mass, m, of batter for cookies at this factory.

Solution

Model the situation by the equation $|m - 55| = 2.5$.

Method 1: Use a Number Line
The absolute value equation $|m - 55| = 2.5$ means that the distance between m and 55 is 2.5 units. To find m on a number line, start at 55 and move 2.5 units in either direction.
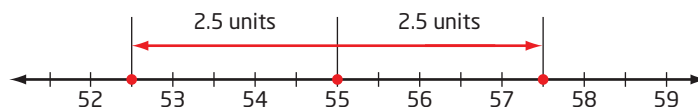
The distance from 55 to 52.5 is 2.5 units.   The distance from 55 to 57.5 is 2.5 units.

The maximum mass is 57.5 g and the minimum mass is 52.5 g.

Method 2: Use an Algebraic Method

Using the definition of absolute value,

$$|m - 55| = \begin{cases} m - 55, \text{ if } m \geq 55 \\ -(m - 55), \text{ if } m < 55 \end{cases}$$

Case 1

$$m - 55 = 2.5$$
$$m = 57.5$$

Case 2

$$-(m - 55) = 2.5$$
$$m - 55 = -2.5$$
$$m = 52.5$$

The maximum mass is 57.5 g and the minimum mass is 52.5 g.

Your Turn

A computerized process controls the amount of fish that is packaged in a specific size of can. The computer program sets the ideal mass at 170 g but allows a tolerance of ± 6 g. Solve an absolute value equation for the maximum and minimum mass, m, of fish in this size of can.

Example 3

Absolute Value Equation With an Extraneous Solution

Solve $|2x - 5| = 5 - 3x$.

Solution

Using the definition of absolute value,

$$|2x - 5| = \begin{cases} 2x - 5, \text{ if } x \geq \dfrac{5}{2} \\ -(2x - 5), \text{ if } x < \dfrac{5}{2} \end{cases}$$

How do you determine the restrictions on the domain in this example?

So, $|2x - 5| = 5 - 3x$ means $2x - 5 = 5 - 3x$ when $x \geq \dfrac{5}{2}$

or $-(2x - 5) = 5 - 3x$ when $x < \dfrac{5}{2}$.

Case 1

$$2x - 5 = 5 - 3x$$
$$5x = 10$$
$$x = 2$$

The value 2 does not satisfy the condition $x \geq \dfrac{5}{2}$, so it is an extraneous solution.

Case 2

$$-(2x - 5) = 5 - 3x$$
$$-2x + 5 = 5 - 3x$$
$$x = 0$$

The value 0 does satisfy the condition $x < \dfrac{5}{2}$.

Verify the solutions.

For $x = 2$:

Left Side	Right Side
$\lvert 2x - 5 \rvert$	$5 - 3x$
$= \lvert 2(2) - 5 \rvert$	$= 5 - 3(2)$
$= \lvert 4 - 5 \rvert$	$= 5 - 6$
$= \lvert -1 \rvert$	$= -1$
$= 1$	

Left Side $\neq$ Right Side

For $x = 0$:

Left Side	Right Side
$\lvert 2x - 5 \rvert$	$5 - 3x$
$= \lvert 2(0) - 5 \rvert$	$= 5 - 3(0)$
$= \lvert 0 - 5 \rvert$	$= 5 - 0$
$= \lvert -5 \rvert$	$= 5$
$= 5$	

Left Side $=$ Right Side

The solution is $x = 0$.

Some absolute value equations may have extraneous roots. Verify potential solutions by substituting them into the original equation.

Your Turn

Solve $\lvert x + 5 \rvert = 4x - 1$.
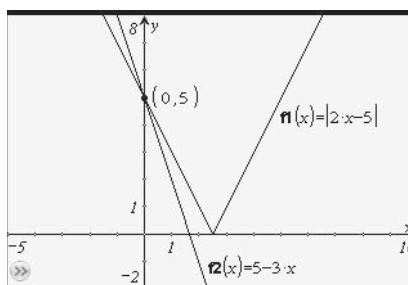
Example 4

Absolute Value Equation With No Solution

Solve $\lvert 3x - 4 \rvert + 12 = 9$.

Solution

$\lvert 3x - 4 \rvert + 12 = 9$ Isolate the absolute value expression.

$\qquad \lvert 3x - 4 \rvert = -3$ This statement is never true.

Since the absolute value of a number is always greater than or equal to zero, by inspection this equation has no solution.

The solution set for this type of equation is the empty set.

Your Turn

Solve $\lvert 4x - 5 \rvert + 9 = 2$.

Did You Know?

The empty set is a set with no elements and is symbolized by {} or ∅.

Example 5

Solve an Absolute Value Equation Involving a Quadratic Expression

Solve $|x^2 - 2x| = 1$.

Solution

Using the definition of absolute value,

$$|x^2 - 2x| = \begin{cases} x^2 - 2x, & \text{if } x \le 0 \text{ or } x \ge 2 \\ -(x^2 - 2x), & \text{if } 0 < x < 2 \end{cases}$$

How can you use the x-intercepts of the related parabola and the direction in which it opens to determine the domain for each case?

Case 1

$$x^2 - 2x = 1$$
$$x^2 - 2x - 1 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Why is the quadratic formula used to solve for x?

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$
$$x = \frac{2 \pm \sqrt{8}}{2}$$
$$x = \frac{2 \pm 2\sqrt{2}}{2}$$
$$x = 1 \pm \sqrt{2}$$

Determine whether $x = 1 + \sqrt{2}$ or $x = 1 - \sqrt{2}$ satisfies the original equation $|x^2 - 2x| = 1$.

For $x = 1 + \sqrt{2}$:

Left Side	Right Side
$\|x^2 - 2x\|$	1
$= \|(1 + \sqrt{2})^2 - 2(1 + \sqrt{2})\|$	
$= \|1 + 2\sqrt{2} + 2 - 2 - 2\sqrt{2}\|$	
$= \|1\|$	
$= 1$	

Left Side = Right Side

For $x = 1 - \sqrt{2}$:

Left Side	Right Side
$\|x^2 - 2x\|$	1
$= \|(1 - \sqrt{2})^2 - 2(1 - \sqrt{2})\|$	
$= \|1 - 2\sqrt{2} + 2 - 2 + 2\sqrt{2}\|$	
$= \|1\|$	
$= 1$	

Left Side = Right Side

Case 2

$$-(x^2 - 2x) = 1$$
$$x^2 - 2x = -1$$
$$x^2 - 2x + 1 = 0$$
$$(x - 1)^2 = 0$$
$$x - 1 = 0$$
$$x = 1$$

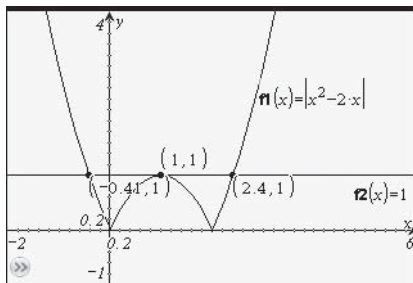Determine whether $x = 1$ satisfies the original equation $|x^2 - 2x| = 1$.

Left Side   Right Side
$|x^2 - 2x|$ 1
$= |1^2 - 2(1)|$
$= |-1|$
$= 1$

  Left Side = Right Side

The solutions are $x = 1$, $x = 1 + \sqrt{2}$, and $x = 1 - \sqrt{2}$.

You can also verify the solution graphically as $x = 1$, $x \approx 2.4$, and $x \approx -0.41$.

Your Turn

Solve $|x^2 - 3x| = 2$.

Example 6

Solve an Absolute Value Equation Involving Linear and Quadratic Expressions

Solve $|x - 10| = x^2 - 10x$.

Solution

Using the definition of absolute value,

$$|x - 10| = \begin{cases} x - 10, & \text{if } x \geq 10 \\ -(x - 10), & \text{if } x < 10 \end{cases}$$

Case 1

$$x - 10 = x^2 - 10x$$
$$0 = x^2 - 11x + 10$$
$$0 = (x - 10)(x - 1)$$
$$x - 10 = 0 \quad \text{or} \quad x - 1 = 0$$
$$x = 10 \qquad\qquad x = 1$$

Only $x = 10$ satisfies the condition $x \geq 10$, so $x = 1$ is an extraneous root.

Case 2

$$-(x - 10) = x^2 - 10x$$
$$-x + 10 = x^2 - 10x$$
$$0 = x^2 - 9x - 10$$
$$0 = (x - 10)(x + 1)$$
$$x - 10 = 0 \quad \text{or} \quad x + 1 = 0$$
$$x = 10 \qquad\qquad x = -1$$

Only $x = -1$ satisfies the condition $x < 10$ for this case. But $x = 10$ satisfies the condition in Case 1, so the solutions are $x = 10$ and $x = -1$.

How could you verify these solutions?

Your Turn

Solve $|x - 5| = x^2 - 8x + 15$.

Key Ideas

- You can solve absolute value equations by graphing the left side and the right side of the equation on the same set of axes and determining the points of intersection.

- To solve an absolute value equation algebraically:
  - Consider the two separate cases, corresponding to the two parts of the definition of absolute value:

    $$|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$$

  - Roots that satisfy the specified condition in each case are solutions to the equation.
  - Identify and reject extraneous roots.

- Verify roots through substitution into the original equation.

- Any absolute value equation of the form $|f(x)| = a$, where $a < 0$, has no solution since by definition $|f(x)| \geq 0$.

Practise

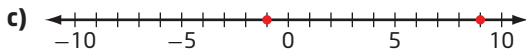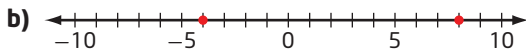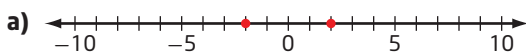**1.** Use the number line to geometrically solve each equation.
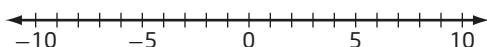
a) $|x| = 7$ **b)** $|x| + 8 = 12$

c) $|x| + 4 = 4$ **d)** $|x| = -6$

2. Solve each absolute value equation by graphing.

a) $|x - 4| = 10$ **b)** $|x + 3| = 2$

c) $6 = |x + 8|$ **d)** $|x + 9| = -3$

3. Determine an absolute value equation in the form $|ax + b| = c$ given its solutions on the number line.

a)

b)

c)

4. Solve each absolute value equation algebraically. Verify your solutions.

a) $|x + 7| = 12$

b) $|3x - 4| + 5 = 7$

c) $2|x + 6| + 12 = -4$

d) $-6|2x - 14| = -42$

5. Solve each equation.

a) $|2a + 7| = a - 4$

b) $|7 + 3x| = 11 - x$

c) $|1 - 2m| = m + 2$

d) $|3x + 3| = 2x - 5$

e) $3|2a + 7| = 3a + 12$

6. Solve each equation and verify your solutions graphically.

a) $|x| = x^2 + x - 3$

b) $|x^2 - 2x + 2| = 3x - 4$

c) $|x^2 - 9| = x^2 - 9$

**d)** $|x^2 - 1| = x$

e) $|x^2 - 2x - 16| = 8$

Apply

7. Bolts are manufactured at a certain factory to have a diameter of 18 mm and are rejected if they differ from this by more than 0.5 mm.

a) Write an absolute value equation in the form $|d - a| = b$ to describe the acceptance limits for the diameter, d, in millimetres, of these bolts, where a and b are real numbers.

b) Solve the resulting absolute value equation to find the maximum and minimum diameters of the bolts.

8. One experiment measured the speed of light as 299 792 456.2 m/s with a measurement uncertainty of 1.1 m/s.

a) Write an absolute value equation in the form $|c - a| = b$ to describe the measured speed of light, c, metres per second, where a and b are real numbers.

b) Solve the absolute value equation to find the maximum and minimum values for the speed of light for this experiment.

9. In communities in Nunavut, aviation fuel is stored in huge tanks at the airport. Fuel is re-supplied by ship yearly. The fuel tank in Kugaaruk holds 50 000 L. The fuel re-supply brings a volume, V, in litres, of fuel plus or minus 2000 L.

a) Write an absolute value equation in the form $|V - a| = b$ to describe the limits for the volume of fuel delivered, where a and b are real numbers.

b) Solve your absolute value equation to find the maximum and minimum volumes of fuel.

10. Consider the statement $x = 7 \pm 4.8$.

    a) Describe the values of x.

    b) Translate the statement into an equation involving absolute value.

11. When measurements are made in science, there is always a degree of error possible. Absolute error is the uncertainty of a measurement. For example, if the mass of an object is known to be 125 g, but the absolute error is said to be ± 4 g, then the measurement could be as high as 129 g and as low as 121 g.

    a) If the mass of a substance is measured once as 64 g and once as 69 g, and the absolute error is ± 2.5 g, what is the actual mass of the substance?

    b) If the volume of a liquid is measured to be 258 mL with an absolute error of ± 7 mL, what are the least and greatest possible measures of the volume?

12. The moon travels in a elliptical orbit around Earth. The distance between Earth and the moon changes as the moon travels in this orbit. The point where the moon's orbit is closest to Earth is called *perigee*, and the point when it is farthest from Earth is called *apogee*. You can use the equation $|d - 381\ 550| = 25\ 150$ to find these distances, d, in kilometres.
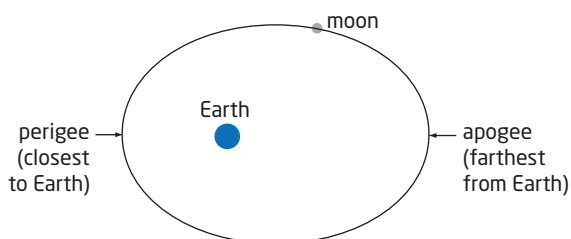
    a) Solve the equation to find the perigee and apogee of the moon's orbit of Earth.

    b) Interpret the given values 381 550 and 25 150 with respect to the distance between Earth and the moon.

Did You Know?

When a full moon is at perigee, it can appear as much as 14% larger to us than a full moon at apogee.
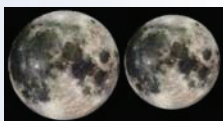
13. Determine whether $n \geq 0$ or $n \leq 0$ makes each equation true.

    a) $n + |-n| = 2n$ **b)** $n + |-n| = 0$

14. Solve each equation for x, where $a, b, c \in \mathrm{R}$.

    a) $|ax| - b = c$ **b)** $|x - b| = c$

15. Erin and Andrea each solve $|x - 4| + 8 = 12$. Who is correct? Explain your reasoning.

Erin's solution:
$$|x - 4| + 8 = 12$$
$$|x - 4| = 4$$
$$x + 4 = 4 \quad \text{or} \quad -x + 4 = 4$$
$$x = 0 \qquad\qquad x = 0$$

Andrea's solution:
$$|x - 4| + 8 = 12$$
$$|x - 4| = 4$$
$$x - 4 = 4 \quad \text{or} \quad -x + 4 = 4$$
$$x = 8 \qquad\qquad x = 0$$

16. Mission Creek in the Okanagan Valley of British Columbia is the site of the spawning of Kokanee salmon every September. Kokanee salmon are sensitive to water temperature. If the water is too cold, egg hatching is delayed, and if the water is too warm, the eggs die. Biologists have found that the spawning rate of the salmon is greatest when the water is at an average temperature of 11.5 °C with an absolute value difference of 2.5 °C. Write and solve an absolute value equation that determines the limits of the ideal temperature range for the Kokanee salmon to spawn.

Did You Know?

In recent years, the September temperature of Mission Creek has been rising. Scientists are considering reducing the temperature of the water by planting more vegetation along the creek banks. This would create shade, cooling the water.

17. Low-dose aspirin contains 81 mg of the active ingredient acetylsalicylic acid (ASA) per tablet. It is used to regulate and reduce heart attack risk associated with high blood pressure by thinning the blood.

a) Given a tolerance of 20% for generic brands, solve an absolute value equation for the maximum and minimum amount of ASA per tablet.

b) Which limit might the drug company tend to lean toward? Why?

Extend

18. For the launch of the Ares I-X rocket from the Kennedy Space Center in Florida in 2009, scientists at NASA indicated they had a launch window of 08:00 to 12:00 eastern time. If a launch at any time in this window is acceptable, write an absolute value equation to express the earliest and latest acceptable times for launch.


19. Determine whether each statement is sometimes true, always true, or never true, where a is a natural number. Explain your reasoning.

a) The value of $|x + 1|$ is greater than zero.

b) The solution to $|x + a| = 0$ is greater than zero.

c) The value of $|x + a| + a$ is greater than zero.

20. Write an absolute value equation with the indicated solutions or type of solution.

a) -2 and 8

b) no solution

c) one integral solution

d) two integral solutions

21. Does the absolute value equation $|ax + b| = 0$, where $a, b \in$ R, always have a solution? Explain.

Create Connections

22. For each graph, an absolute value function and a linear function intersect to produce solutions to an equation composed of the two functions. Determine the equation that is being solved in each graph.

a)

b)
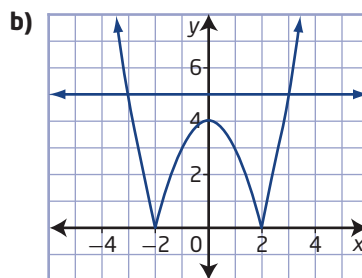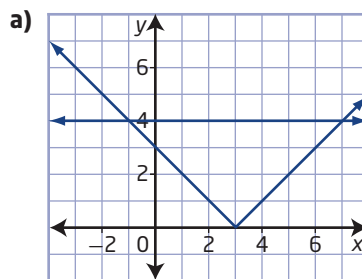
23. Explain, without solving, why the equation $|3x + 1| = -2$ has no solutions, while the equation $|3x + 1| - 4 = -2$ has solutions.

24. Why do some absolute value equations produce extraneous roots when solved algebraically? How are these roots created in the algebraic process if they are not actual solutions of the equation?

7.4

Reciprocal Functions

Focus on...

- graphing the reciprocal of a given function
- analysing the graph of the reciprocal of a given function
- comparing the graph of a function to the graph of the reciprocal of that function
- identifying the values of x for which the graph of $y = \dfrac{1}{f(x)}$ has vertical asymptotes

Isaac Newton (1643–1727) is one of the most important mathematicians and physicists in history. Besides being the co-inventor of calculus, Newton is famous for deriving the law of universal gravitation. He deduced that the forces that keep the planets in their orbits must be related reciprocally as the squares of their distances from the centres about which they revolve.

$$F_1 = F_2 = G\left(\frac{m_1 \times m_2}{r^2}\right)$$
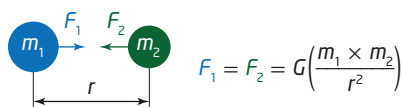
As a result of the reciprocal relationship, as the distance, r, between two planets increases, the gravitational force, F, decreases. Similarly, as the distance decreases, the gravitational force between the planets will increase.

Did You Know?

Isaac Newton made most of his important discoveries in the 1660s. During this time he was forced to work at home because the bubonic plague resulted in the closure of all public buildings, including Cambridge University, where Newton studied.

Web **Link**

To learn more about Isaac Newton and his contributions to mathematics, go to www.mhrprecalc11.ca and follow the links.

Investigate Exchange Rates

Materials

- graphing calculator

Perhaps you have travelled to Mexico or to Hawaii and have exchanged Canadian dollars for pesos or U.S. dollars. Perhaps you have travelled overseas and exchanged British pounds for the Japanese yen or Swiss franc. If so, you have experienced exchange rates in action. Do you know how they work?

An exchange rate is the rate at which one currency is converted into another currency. Exchange rates are typically quoted as a ratio with either one of the currencies being set equal to one, such as 1 Australian dollar = 0.9796 Canadian dollars.

1. If the Canadian dollar is worth US$0.80, it costs C$1.25 to buy US$1. Change the values 0.80 and 1.25 to fractions in lowest terms. Can you see how these fractions are related to each other? Discuss with your classmates how you could use this relationship to determine exchange rates.

2. In step 1, the Canadian-to-U.S. dollar exchange rate is 0.80. What is the U.S.-to-Canada dollar exchange rate? How many Canadian dollars could you buy with US$1?

3. a) Copy and complete the table to determine the purchase price of US$1 for various Canadian-to-U.S. dollar exchange rates.

b) Describe your method of determining the purchase price of US$1. Would your method work for all currency exchanges?

c) Plot the ordered pairs from the table of values. Draw a smooth curve through the points. Extrapolate. Does this curve have an x-intercept? a y-intercept? Explain.

C$1 in US$	Purchase Price of US$1
0.65	1.54
0.70	
0.75	
0.80	
0.85	
0.90	
0.95	
1.00	
1.05	
1.10	

4. Examine the currency exchange table shown. The Japanese yen (¥) is shown as 0.0108. What does this number represent in terms of exchange rates?

CURRENCIES

Currency	In C$	Currency	In C$	Currency	In C$
Australia dollar	0.9796	Euro	1.5748	Peru sol	0.3629
Bahamas dollar	1.0525	Hong Kong dollar	0.1346	Philippine peso	0.0216
Bahrain dinar	2.7944	India rupee	0.0219	Poland zloty	0.3749
Barbados dollar	0.5287	Jamaica dollar	0.0113	Russia rouble	0.0346
Brazil real	0.6134	Japan yen	0.0108	Singapore dollar	0.7571
Chile peso	0.0020	Kenya shilling	0.0143	South Africa rand	0.1417
Chinese yuan	0.1534	S. Korea won	0.0009	Switzerland franc	1.0418
Denmark krone	0.2107	Mexico peso	0.0786	Ukraine hryvna	0.1238
Dominican peso	0.0286	New Zealand dollar	0.7801	U.A.E. dirham	0.2842
Egypt pound	0.1924	Pakistan rupee	0.0120	U.K. pound	1.7266
				U.S. dollar	1.0374

5. a) How many yen can you purchase with C$1? With C$200?

b) How much does it cost to purchase ¥5000?

6. Choose one other currency from the table or find current currency exchange rates on the Internet.

a) How much of that currency can be purchased with C$1?

b) How much does it cost to purchase 100 units of the foreign currency?

Reflect and Respond

7. Analyse the relationship of currency exchange between countries. For example, when you have the Canadian-to-U.S. dollar exchange rate, how do you determine the U.S.-to-Canadian dollar exchange rate? What is the relationship between the two calculations?

8. Does the relationship in step 7 always work?

Recall that the product of a number and its reciprocal is always equal to 1. For example, $\frac{3}{4}$ is the reciprocal of $\frac{4}{3}$ and $\frac{4}{3}$ is the reciprocal of $\frac{3}{4}$ because $\frac{3}{4}\left(\frac{4}{3}\right) = 1$.

So, for any non-zero real number a, the reciprocal of a is $\frac{1}{a}$ and the reciprocal of $\frac{1}{a}$ is a. For a function $f(x)$, its reciprocal is $\frac{1}{f(x)}$, provided that $f(x) \neq 0$.

Example 1

Compare the Graphs of a Function and Its Reciprocal

reciprocal function

- a function $y = \frac{1}{f(x)}$ defined by $y = \frac{1}{f(a)} = \frac{1}{b}$ if $f(a) = b$, $f(a) \neq 0$, $b \neq 0$

Sketch the graphs of $y = f(x)$ and its **reciprocal function** $y = \frac{1}{f(x)}$, where $f(x) = x$. Examine how the functions are related.

Solution

Use a table of values to graph the functions $y = x$ and $y = \frac{1}{x}$.

x	$y = x$	$y = \frac{1}{x}$
-10	-10	$-\frac{1}{10}$
-5	-5	$-\frac{1}{5}$
-2	-2	$-\frac{1}{2}$
-1	-1	-1
$-\frac{1}{2}$	$-\frac{1}{2}$	-2
$-\frac{1}{5}$	$-\frac{1}{5}$	-5
$-\frac{1}{10}$	$-\frac{1}{10}$	-10
0	0	undefined
$\frac{1}{10}$	$\frac{1}{10}$	10
$\frac{1}{5}$	$\frac{1}{5}$	5
$\frac{1}{2}$	$\frac{1}{2}$	2
1	1	1
2	2	$\frac{1}{2}$
5	5	$\frac{1}{5}$
10	10	$\frac{1}{10}$

Notice that the function values for $y = \frac{1}{x}$ can be found by taking the reciprocal of the function values for $y = x$.

What is unique about the reciprocals of -1 and 1? Why?

Why is the reciprocal of 0 undefined?

What happens to the value of the reciprocal as the absolute value of a number increases in value?

The function $y = x$ is a function of degree one, so its graph is a line.

The function $y = \dfrac{1}{x}$ is a rational function.

Its graph has two distinct pieces, or branches. These branches are located on either side of the **vertical asymptote**, defined by the non-permissible value of the domain of the rational function, and the **horizontal asymptote**, defined by the fact that the value 0 is not in the range of the function.

Characteristic	$y = x$	$y = \dfrac{1}{x}$
Domain	$\{x \mid x \in R\}$	$\{x \mid x \neq 0, x \in R\}$
Range	$\{y \mid y \in R\}$	$\{y \mid y \neq 0, y \in R\}$
End behaviour	• If $x > 0$ and $\lvert x \rvert$ is very large, then $y > 0$ and is very large. • If $x < 0$ and $\lvert x \rvert$ is very large, then $y < 0$ and $\lvert y \rvert$ is very large.	• If $x > 0$ and $\lvert x \rvert$ is very large, then $y > 0$ and is close to 0. • If $x < 0$ and $\lvert x \rvert$ is very large, then $y < 0$ and y is close to 0.
Behaviour at $x = 0$	$y = 0$	undefined, vertical asymptote at $x = 0$
Invariant points	$(-1, -1)$ and $(1, 1)$	

Why does the curve approach the y-axis but never touch it?

Why does the curve approach the x-axis but never touch it?

asymptote
• a line whose distance from a given curve approaches zero

vertical asymptote
• for reciprocal functions, occur at the non-permissible values of the function
• the line $x = a$ is a vertical asymptote if the curve approaches the line more and more closely as x approaches a, and the values of the function increase or decrease without bound as x approaches a

horizontal asymptote
• describes the behaviour of a graph when $\lvert x \rvert$ is very large
• the line $y = b$ is a horizontal asymptote if the values of the function approach b when $\lvert x \rvert$ is very large

Create a table of values and sketch the graphs of $y = f(x)$ and its reciprocal $y = \dfrac{1}{f(x)}$, where $f(x) = -x$. Examine how the functions are related.

Example 2

Graph the Reciprocal of a Linear Function

Consider $f(x) = 2x + 5$.

a) Determine its reciprocal function $y = \dfrac{1}{f(x)}$.

b) Determine the equation of the vertical asymptote of the reciprocal function.

c) Graph the function $y = f(x)$ and its reciprocal function $y = \dfrac{1}{f(x)}$.

Describe a strategy that could be used to sketch the graph of a reciprocal function.

Solution

a) The reciprocal function is $y = \dfrac{1}{2x + 5}$.

b) A vertical asymptote occurs at any non-permissible values of the corresponding rational expression $\dfrac{1}{2x + 5}$.

To determine non-permissible values, set the denominator equal to 0 and solve.

$$2x + 5 = 0$$
$$2x = -5$$
$$x = -\frac{5}{2}$$

How are the zeros of the function $f(x) = 2x + 5$ related to the vertical asymptotes of its reciprocal function $y = \dfrac{1}{2x + 5}$?

The non-permissible value is $x = -\dfrac{5}{2}$.

In the domain of the rational expression $\dfrac{1}{2x + 5}$, $x \neq -\dfrac{5}{2}$.

The reciprocal function is undefined at this value, and its graph has a vertical asymptote with equation $x = -\dfrac{5}{2}$.

c) Method 1: Use Pencil and Paper

To sketch the graph of the function $f(x) = 2x + 5$, use the y-intercept of 5 and slope of 2.

To sketch the graph of the reciprocal of a function, consider the following characteristics:

Characteristic	Function $f(x) = 2x + 5$	Reciprocal Function $f(x) = \dfrac{1}{2x + 5}$
x-intercept and asymptotes	• The value of the function is zero at $x = -\dfrac{5}{2}$.	• The value of the reciprocal function is undefined at $x = -\dfrac{5}{2}$. A vertical asymptote exists.
Invariant points	• Solve $2x + 5 = 1$. • The value of the function is $+1$ at $(-2, 1)$.	• Solve $\dfrac{1}{2x + 5} = 1$. • The value of the reciprocal function is $+1$ at $(-2, 1)$.
	• Solve $2x + 5 = -1$. • The value of the function is -1 at $(-3, -1)$.	• Solve $\dfrac{1}{2x + 5} = -1$. • The value of the reciprocal function is -1 at $(-3, -1)$.

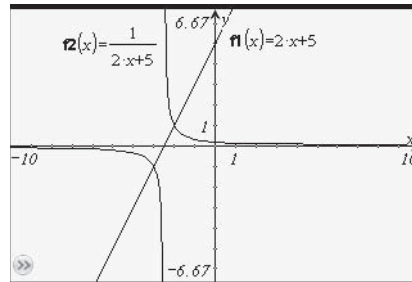The graphs of $y = 2x + 5$ and $y = \dfrac{1}{2x + 5}$ are shown.
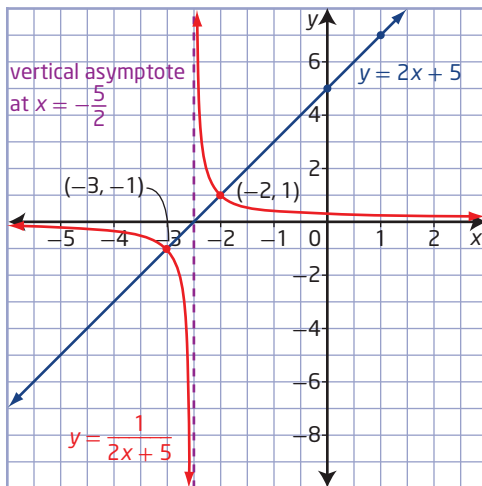
Method 2: Use a Graphing Calculator

Graph the functions using a graphing calculator.

Enter the functions as $y = 2x + 5$ and $y = \dfrac{1}{2x + 5}$ or as $y = f(x)$

and $y = \dfrac{1}{f(x)}$, where $f(x)$ has been defined as $f(x) = 2x + 5$.

Ensure that both branches of the reciprocal function are visible.

How can you determine if the window settings you chose are the most appropriate?

What are the asymptotes? How do you know?

Use the calculator's value and zero features to verify the invariant points and the y-intercept.

Use the table feature on the calculator
- to see the nature of the ordered pairs that exist when a function and its reciprocal are graphed
- to compare the two functions in terms of values remaining positive or negative or values of y increasing or decreasing
- to see what happens to the reciprocal function as the absolute values of x get very large or very small
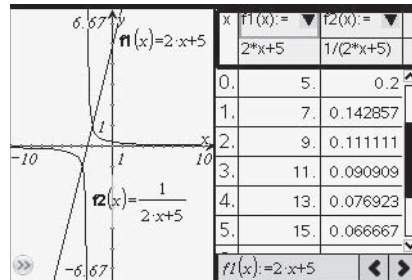
Your Turn

Consider $f(x) = 3x - 9$.

a) Determine its reciprocal function $y = \dfrac{1}{f(x)}$.

b) Determine the equation of the vertical asymptote of the reciprocal function.

c) Graph the function $y = f(x)$ and its reciprocal function $y = \dfrac{1}{f(x)}$, with and without technology. Discuss the behaviour of $y = \dfrac{1}{f(x)}$ as it nears its asymptotes.

Example 3

Graph the Reciprocal of a Quadratic Function

Consider $f(x) = x^2 - 4$.
a) What is the reciprocal function of $f(x)$?
b) State the non-permissible values of x and the equation(s) of the vertical asymptote(s) of the reciprocal function.
c) What are the x-intercepts and the y-intercept of the reciprocal function?
d) Graph the function $y = f(x)$ and its reciprocal function $y = \dfrac{1}{f(x)}$.

Solution

a) The reciprocal function is $y = \dfrac{1}{x^2 - 4}$.

b) Non-permissible values of x occur when the denominator of the corresponding rational expression is equal to 0.

$$x^2 - 4 = 0$$
$$(x - 2)(x + 2) = 0$$
$$x - 2 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = 2 \qquad\qquad x = -2$$

The non-permissible values of the corresponding rational expression are $x = 2$ and $x = -2$.

The reciprocal function is undefined at these values, so its graph has vertical asymptotes with equations $x = 2$ and $x = -2$.

c) To find the x-intercepts of the function $y = \dfrac{1}{x^2 - 4}$, let $y = 0$.

$$0 = \frac{1}{x^2 - 4}$$

There is no value of x that makes this equation true. Therefore, there are no x-intercepts.

To find the y-intercept, substitute 0 for x.

$$y = \frac{1}{0^2 - 4}$$
$$y = -\frac{1}{4}$$

The y-intercept is $-\dfrac{1}{4}$.

d) **Method 1: Use Pencil and Paper**
For $f(x) = x^2 - 4$, the coordinates of the vertex are $(0, -4)$.
The x-intercepts occur at $(-2, 0)$ and $(2, 0)$.

Use this information to plot the graph of $f(x)$.

To sketch the graph of the reciprocal function,
• Draw the asymptotes.
• Plot the invariant points where $f(x) = \pm 1$. The exact locations of the invariant points can be found by solving $x^2 - 4 = \pm 1$.

Solving $x^2 - 4 = 1$ results in the points $(\sqrt{5}, 1)$ and $(-\sqrt{5}, 1)$.
Solving $x^2 - 4 = -1$ results in the points $(\sqrt{3}, -1)$ and $(-\sqrt{3}, -1)$.

• The y-coordinates of the points on the graph of the reciprocal function are the reciprocals of the y-coordinates of the corresponding points on the graph of $f(x)$.
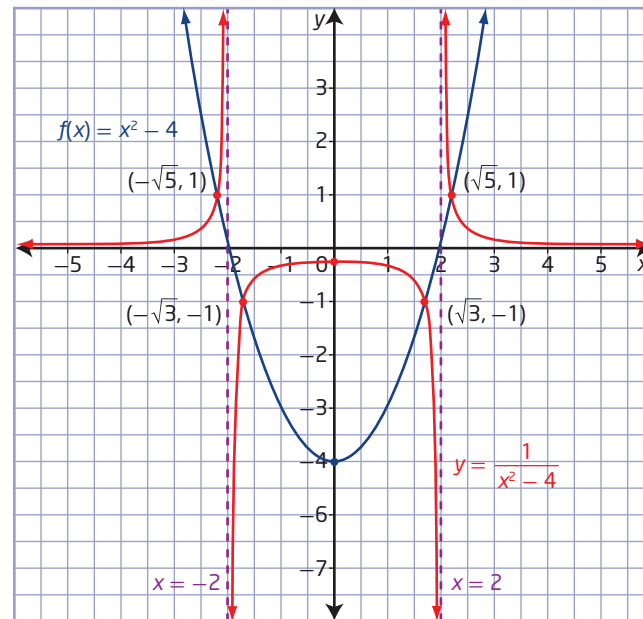
Method 2: Use a Graphing Calculator

Enter the functions $y = x^2 - 4$ and $y = \dfrac{1}{x^2 - 4}$.

Adjust the window settings so that the vertex and intercepts of $y = x^2 - 4$ are visible, if necessary.
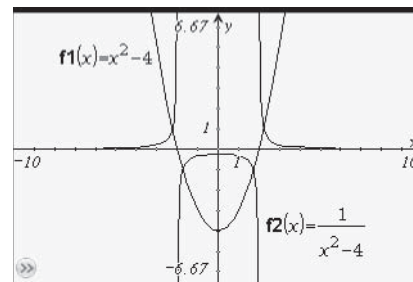
Your Turn

Consider $f(x) = x^2 + x - 6$.

a) What is the reciprocal function of $f(x)$?

b) State the non-permissible values of x and the equation(s) of the vertical asymptote(s) of the reciprocal function.

c) What are the x-intercepts and the y-intercept of the reciprocal function?

d) Sketch the graphs of $y = f(x)$ and its reciprocal function $y = \dfrac{1}{f(x)}$.

Example 4

Graph $y = f(x)$ Given the Graph of $y = \dfrac{1}{f(x)}$

The graph of a reciprocal function of the form $y = \dfrac{1}{ax + b}$, where a and b are non-zero constants, is shown.

a) Sketch the graph of the original function, $y = f(x)$.

b) Determine the original function, $y = f(x)$.
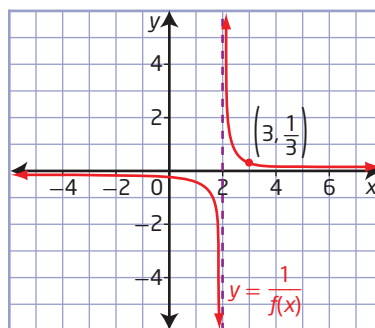
Solution

a) Since $y = \dfrac{1}{f(x)} = \dfrac{1}{ax + b}$, the original function is of the form $f(x) = ax + b$, which is a linear function. The reciprocal graph has a vertical asymptote at $x = 2$, so the graph of $y = f(x)$ has an x-intercept at $(2, 0)$. Since $\left(3, \dfrac{1}{3}\right)$ is a point on the graph of $y = \dfrac{1}{f(x)}$, the point $(3, 3)$ must be on the graph of $y = f(x)$.

Draw a line passing through $(2, 0)$ and $(3, 3)$.
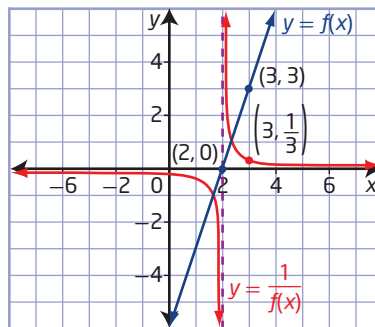
b) Method 1: Use the Slope and the y-Intercept

Write the function in the form $y = mx + b$. Use the coordinates of the two known points, $(2, 0)$ and $(3, 3)$, to determine that the slope, m, is 3. Substitute the coordinates of one of the points into $y = 3x + b$ and solve for b.

$b = -6$

The original function is $f(x) = 3x - 6$.

Method 2: Use the x-Intercept

With an x-intercept of 2, the function $f(x)$ is based on the factor $x - 2$, but it could be a multiple of that factor.

$$f(x) = a(x - 2)$$

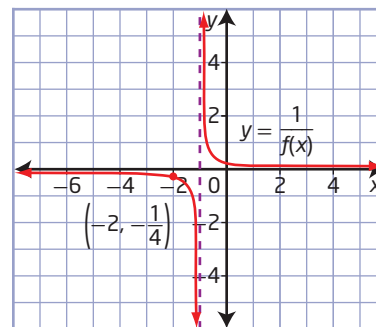Use the point (3, 3) to find the value of a.

$$3 = a(3 - 2)$$
$$3 = a$$

The original function is $f(x) = 3(x - 2)$, or $f(x) = 3x - 6$.

Your Turn

The graph of a reciprocal function of the form $y = \dfrac{1}{f(x)} = \dfrac{1}{ax + b}$, where a and b are non-zero constants, is shown.

a) Sketch the graph of the original function, $y = f(x)$.

b) Determine the original function, $y = f(x)$.

Key Ideas

- If $f(x) = x$, then $\dfrac{1}{f(x)} = \dfrac{1}{x}$, where $\dfrac{1}{f(x)}$ denotes a reciprocal function.

- You can obtain the graph of $y = \dfrac{1}{f(x)}$ from the graph of $y = f(x)$ by using the following guidelines:

  - The non-permissible values of the reciprocal function are related to the position of the vertical asymptotes. These are also the non-permissible values of the corresponding rational expression, where the reciprocal function is undefined.

  - Invariant points occur when the function $f(x)$ has a value of 1 or -1. To determine the x-coordinates of the invariant points, solve the equations $f(x) = \pm 1$.

  - The y-coordinates of the points on the graph of the reciprocal function are the reciprocals of the y-coordinates of the corresponding points on the graph of $y = f(x)$.

  - As the value of x approaches a non-permissible value, the absolute value of the reciprocal function gets very large.

  - As the absolute value of x gets very large, the absolute value of the reciprocal function approaches zero.

- The domain of the reciprocal function is the same as the domain of the original function, excluding the non-permissible values.
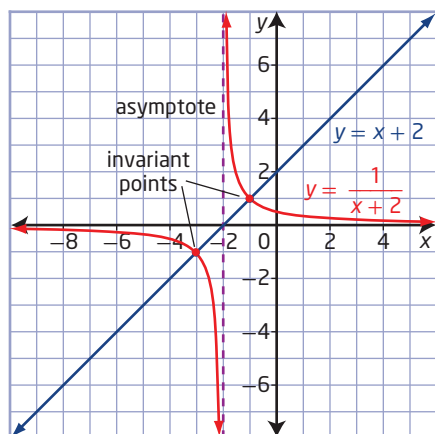
Check Your Understanding

Practise

1. Given the function $y = f(x)$, write the corresponding reciprocal function.

a) $y = -x + 2$

b) $y = 3x - 5$

c) $y = x^2 - 9$

d) $y = x^2 - 7x + 10$

2. For each function,

i) state the zeros

ii) write the reciprocal function

iii) state the non-permissible values of the corresponding rational expression

iv) explain how the zeros of the original function are related to the non-permissible values of the reciprocal function

v) state the equation(s) of the vertical asymptote(s)

a) $f(x) = x + 5$ **b)** $g(x) = 2x + 1$

c) $h(x) = x^2 - 16$ **d)** $t(x) = x^2 + x - 12$

3. State the equation(s) of the vertical asymptote(s) for each function.

   a) $f(x) = \dfrac{1}{5x - 10}$

   b) $f(x) = \dfrac{1}{3x + 7}$

   c) $f(x) = \dfrac{1}{(x - 2)(x + 4)}$

   d) $f(x) = \dfrac{1}{x^2 - 9x + 20}$

4. The calculator screen gives a function table for $f(x) = \dfrac{1}{x - 3}$. Explain why there is an undefined statement.

x	f1(x):= ▼
	1/(x−3)
0.	−0.3333...
1.	−0.5
2.	−1.
3.	#UNDEF
4.	1.

5. What are the *x*-intercept(s) and the *y*-intercept of each function?

   a) $f(x) = \dfrac{1}{x + 5}$
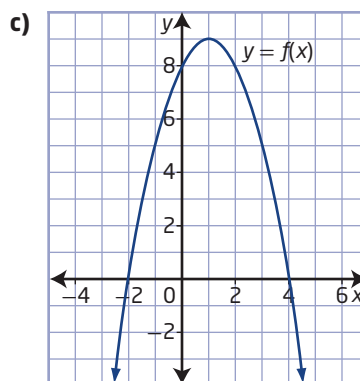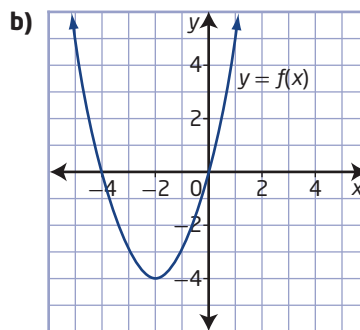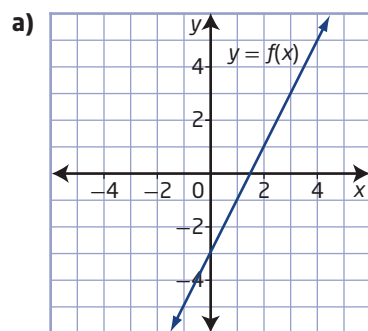
   b) $f(x) = \dfrac{1}{3x - 4}$

   c) $f(x) = \dfrac{1}{x^2 - 9}$

   d) $f(x) = \dfrac{1}{x^2 + 7x + 12}$

6. Copy each graph of $y = f(x)$, and sketch the graph of the reciprocal function $y = \dfrac{1}{f(x)}$. Describe your method.

   a)

b)

c)

7. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes. Label the asymptotes, the invariant points, and the intercepts.

   a) $f(x) = x - 16$

   b) $f(x) = 2x + 4$

   c) $f(x) = 2x - 6$

   d) $f(x) = x - 1$

8. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes. Label the asymptotes, the invariant points, and the intercepts.
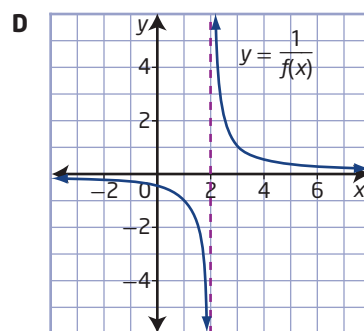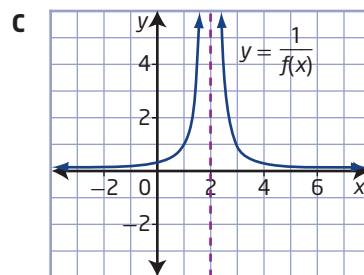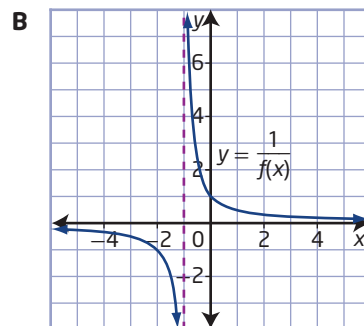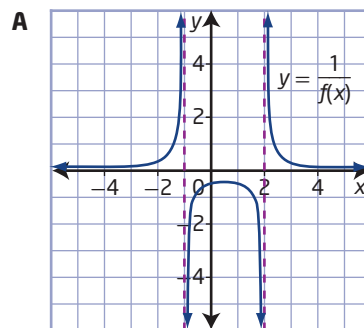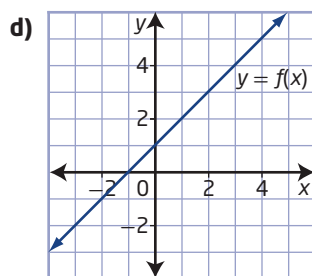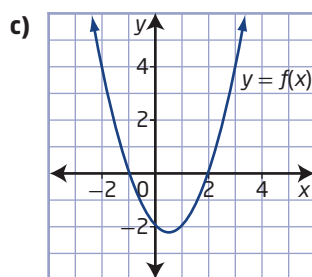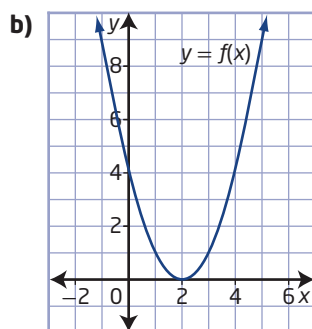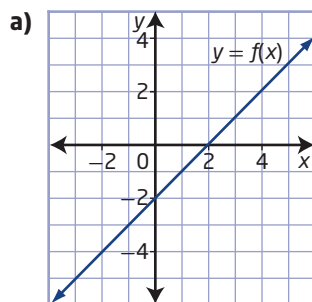
   a) $f(x) = x^2 - 16$

   b) $f(x) = x^2 - 2x - 8$

   c) $f(x) = x^2 - x - 2$

   d) $f(x) = x^2 + 2$

9. Match the graph of the function with the graph of its reciprocal.

a)

b)

c)

d)

A

B

C

D

Apply

10. Each of the following is the graph of a reciprocal function, $y = \dfrac{1}{f(x)}$.

  i) Sketch the graph of the original function, $y = f(x)$.

  ii) Explain the strategies you used.

  iii) What is the original function, $y = f(x)$?

a)

b)
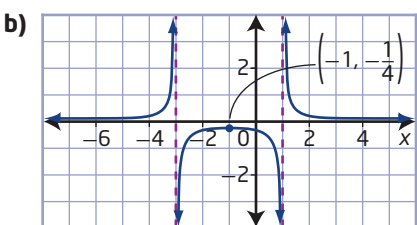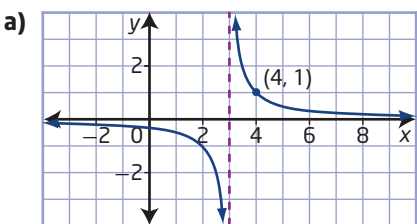
11. You can model the swinging motion of a pendulum using many mathematical rules. For example, the frequency, f, or number of vibrations per second of one swing, in hertz (Hz), equals the reciprocal of the period, T, in seconds, of the swing. The formula is $f = \dfrac{1}{T}$.

  a) Sketch the graph of the function $f = \dfrac{1}{T}$.

  b) What is the reciprocal function?

  c) Determine the frequency of a pendulum with a period of 2.5 s.

  d) What is the period of a pendulum with a frequency of 1.6 Hz?
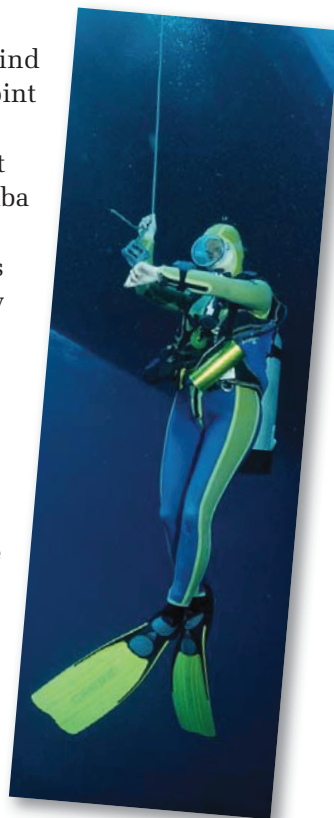
Did You Know?

Much of the mathematics of pendulum motion was described by Galileo, based on his curiosity about a swinging lamp in the Cathedral of Pisa, Italy. His work led to much more accurate measurement of time on clocks.

12. The greatest amount of time, t, in minutes, that a scuba diver can take to rise toward the water surface without stopping for decompression is defined by the function $t = \dfrac{525}{d - 10}$, where d is the depth, in metres, of the diver.

  a) Graph the function using graphing technology.

  b) Determine a suitable domain which represents this application.

  c) Determine the maximum time without stopping for a scuba diver who is 40 m deep.

  d) Graph a second function, $t = 40$. Find the intersection point of the two graphs. Interpret this point in terms of the scuba diver rising to the surface. Check this result algebraically with the original function.

  e) Does this graph have a horizontal asymptote? What does this mean with respect to the scuba diver?

Did You Know?

If scuba divers rise to the water surface too quickly, they may experience decompression sickness or *the bends*, which is caused by breathing nitrogen or other gases under pressure. The nitrogen bubbles are released into the bloodstream and obstruct blood flow, causing joint pain.

13. The pitch, p, in hertz (Hz), of a musical note is the reciprocal of the period, P, in seconds, of the sound wave for that note created by the air vibrations.

a) Write a function for pitch, p, in terms of period, P.

b) Sketch the graph of the function.

c) What is the pitch, to the nearest 0.1 Hz, for a musical note with period 0.048 s?

14. The intensity, I, in watts per square metre (W/m²), of a sound equals 0.004 multiplied by the reciprocal of the square of the distance, d, in metres, from the source of the sound.

a) Write a function for I in terms of d to represent this relationship.

b) Graph this function for a domain of $d > 0$.

c) What is the intensity of a car horn for a person standing 5 m from the car?

15. a) Describe how to find the vertex of the parabola defined by $f(x) = x^2 - 6x - 7$.

b) Explain how knowing the vertex in part a) would help you to graph the function $g(x) = \dfrac{1}{x^2 - 6x - 7}$.

c) Sketch the graph of $g(x) = \dfrac{1}{x^2 - 6x - 7}$.

16. The amount of time, t, to complete a large job is proportional to the reciprocal of the number of workers, n, on the job. This can be expressed as $t = k\left(\dfrac{1}{n}\right)$ or $t = \dfrac{k}{n}$, where k is a constant. For example, the Spiral Tunnels built by the Canadian Pacific Railroad in Kicking Horse Pass, British Columbia, were a major engineering feat when they opened in 1909. Building two spiral tracks each about 1 km long required 1000 workers to work about 720 days. Suppose that each worker performed a similar type of work.

a) Substitute the given values of t and n into the formula to find the constant k.

b) Use technology to graph the function $t = \dfrac{k}{n}$.

c) How much time would have been required to complete the Spiral Tunnels if only 400 workers were on the job?

d) Determine the number of workers needed if the job was to be completed in 500 days.

Did You Know?

Kicking Horse Pass is in Yoho National Park. *Yoho* is a Cree word meaning great awe or astonishment. This may be a reference to the soaring peaks, the rock walls, and the spectacular Takakkaw Falls nearby.

Extend

17. Use the summary of information to produce the graphs of both $y = f(x)$ and $y = \dfrac{1}{f(x)}$, given that $f(x)$ is a linear function.

Interval of x	$x < 3$	$x > 3$
Sign of $f(x)$	+	−
Direction of $f(x)$	decreasing	decreasing
Sign of $\dfrac{1}{f(x)}$	+	−
Direction of $\dfrac{1}{f(x)}$	increasing	increasing

18. Determine whether each statement is true or false, and explain your reasoning.

   a) The graph of $y = \dfrac{1}{f(x)}$ always has a vertical asymptote.

   b) A function in the form of $y = \dfrac{1}{f(x)}$ always has at least one value for which it is not defined.

   c) The domain of $y = \dfrac{1}{f(x)}$ is always the same as the domain of $y = f(x)$.

Create Connections

19. Rita and Jerry are discussing how to determine the asymptotes of the reciprocal of a given function. Rita concludes that you can determine the roots of the corresponding equation, and those values will lead to the equations of the asymptotes. Jerry assumes that when the function is written in rational form, you can determine the non-permissible values. The non-permissible values will lead to the equations of the asymptotes.

   a) Which student has made a correct assumption? Explain your choice.

   b) Is this true for both a linear and a quadratic function?

20. The diagram shows how an object forms an inverted image on the opposite side of a convex lens, as in many cameras. Scientists discovered the relationship
$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$$
where u is the distance from the object to the lens, v is the distance from the lens to the image, and f is the focal length of the lens being used.
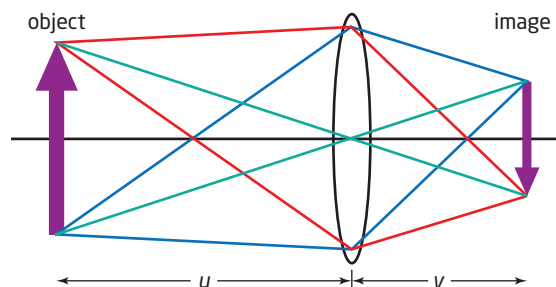
   a) Determine the distance, v, between the lens and the image if the distance, u, to the object is 300 mm and the lens has a focal length, f, of 50 mm.

   b) Determine the focal length of a zoom lens if an object 10 000 mm away produces an inverted image 210 mm behind the lens.

21. **MINI LAB** Use technology to explore the behaviour of a graph near the vertical asymptote and the end behaviour of the graph.

   Consider the function
   $$f(x) = \frac{1}{4x - 2}, \; x \neq \frac{1}{2}.$$

   Step 1 Sketch the graph of the function
   $$f(x) = \frac{1}{4x - 2}, \; x \neq \frac{1}{2}, \text{ drawing in}$$
   the vertical asymptote.

Step 2 a) Copy and complete the tables to show the behaviour of the function as $x \to \left(\frac{1}{2}\right)^{-}$ and as $x \to \left(\frac{1}{2}\right)^{+}$, meaning when x approaches $\frac{1}{2}$ from the left ($-$) and from the right ($+$).

As $x \to \left(\frac{1}{2}\right)^{-}$:

x	f(x)
0	−0.5
0.4	
0.45	
0.47	
0.49	
0.495	
0.499	

As $x \to \left(\frac{1}{2}\right)^{+}$:

x	f(x)
1	0.5
0.6	
0.55	
0.53	
0.51	
0.505	
0.501	

b) Describe the behaviour of the function as the value of x approaches the asymptote. Will this always happen?

Step 3 a) To explore the end behaviour of the function, the absolute value of x is made larger and larger. Copy and complete the tables for values of x that are farther and farther from zero.
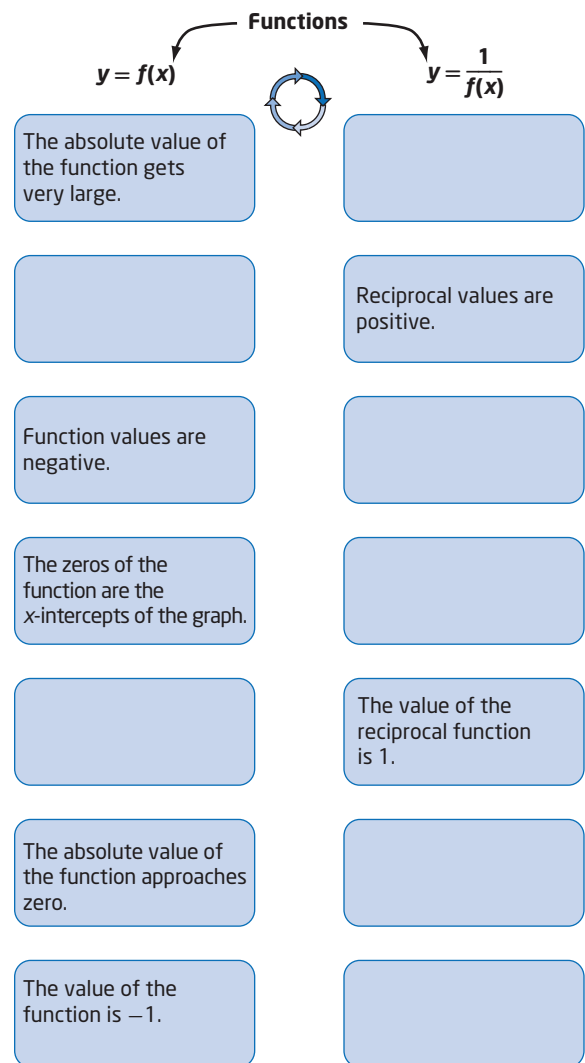
As x becomes smaller:

x	f(x)
−10	$-\frac{1}{42}$
−100	
−1000	
−10 000	
−100 000	

As x becomes larger:

x	f(x)
10	$\frac{1}{38}$
100	
1 000	
10 000	
100 000	

b) Describe what happens to the graph of the reciprocal function as $|x|$ becomes very large.

22. Copy and complete the flowchart to describe the relationship between a function and its corresponding reciprocal function.

Functions

$y = f(x)$ $y = \dfrac{1}{f(x)}$

The absolute value of the function gets very large.	
	Reciprocal values are positive.
Function values are negative.	
The zeros of the function are the x-intercepts of the graph.	
	The value of the reciprocal function is 1.
The absolute value of the function approaches zero.	
The value of the function is −1.	

7.1 Absolute Value, pages 358–367

1. Evaluate.

  a) $|-5|$ **b)** $\left|2\frac{3}{4}\right|$ **c)** $|-6.7|$

2. Rearrange these numbers in order from least to greatest.

$-4,\ \sqrt{9},\ |-3.5|,\ -2.7,\ \left|-\frac{9}{2}\right|,\ |-1.6|,\ \left|1\frac{1}{2}\right|$

3. Evaluate each expression.

  a) $|-7-2|$

  b) $|-3+11-6|$

  c) $5|-3.75|$

  d) $|5^2-7|+|-10+2^3|$

4. A school group travels to Mt. Robson Provincial Park in British Columbia to hike the Berg Lake Trail. From the Robson River bridge, kilometre 0.0, they hike to Kinney Lake, kilometre 4.2, where they stop for lunch. They then trek across the suspension bridge to the campground, kilometre 10.5. The next day they hike to the shore of Berg Lake and camp, kilometre 19.6. On day three, they hike to the Alberta/British Columbia border, kilometre 21.9, and turn around and return to the campground near Emperor Falls, kilometre 15.0. On the final day, they walk back out to the trailhead, kilometre 0.0. What total distance did the school group hike?

5. Over the course of five weekdays, one mining stock on the Toronto Stock Exchange (TSX) closed at $4.28 on Monday, closed higher at $5.17 on Tuesday, finished Wednesday at $4.79, and shot up to close at $7.15 on Thursday, only to finish the week at $6.40.

  a) What is the net change in the closing value of this stock for the week?

  b) Determine the total change in the closing value of the stock.

7.2 Absolute Value Functions, pages 368–379

6. Consider the functions $f(x) = 5x + 2$ and $g(x) = |5x + 2|$.

  a) Create a table of values for each function, using values of $-2, -1, 0, 1,$ and 2 for x.

  b) Plot the points and sketch the graphs of the functions on the same coordinate grid.

  c) Determine the domain and range for both $f(x)$ and $g(x)$.

  d) List the similarities and the differences between the two functions and their corresponding graphs.

7. Consider the functions $f(x) = 8 - x^2$ and $g(x) = |8 - x^2|$.

  a) Create a table of values for each function, using values of $-2, -1, 0, 1,$ and 2 for x.
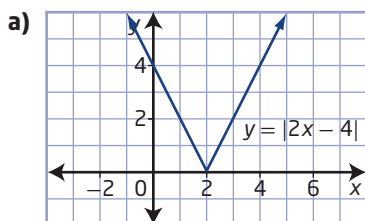
  b) Plot the points and sketch the graphs of the functions on the same coordinate grid.

  c) Determine the domain and range for both $f(x)$ and $g(x)$.

  d) List the similarities and the differences between the two functions and their corresponding graphs.

8. Write the piecewise function that represents each graph.

a)

$y = |2x - 4|$

b)

$y = |x^2 - 1|$
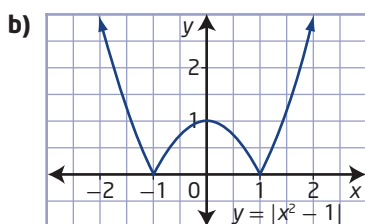
9. a) Explain why the functions $f(x) = 3x^2 + 7x + 2$ and $g(x) = |3x^2 + 7x + 2|$ have different graphs.

b) Explain why the functions $f(x) = 3x^2 + 4x + 2$ and $g(x) = |3x^2 + 4x + 2|$ have identical graphs.

10. An absolute value function has the form $f(x) = |ax + b|$, where $a \neq 0$, $b \neq 0$, and $a, b \in R$. If the function $f(x)$ has a domain of $\{x \mid x \in R\}$, a range of $\{y \mid y \geq 0, y \in R\}$, an x-intercept occurring at $\left(-\frac{2}{3}, 0\right)$, and a y-intercept occurring at $(0, 10)$, what are the values of a and b?

7.3 Absolute Value Equations, pages 380–391

11. Solve each absolute value equation graphically. Express answers to the nearest tenth, when necessary.

a) $|2x - 2| = 9$

b) $|7 + 3x| = x - 1$

c) $|x^2 - 6| = 3$

d) $|m^2 - 4m| = 5$

12. Solve each equation algebraically.

a) $|q + 9| = 2$

b) $|7x - 3| = x + 1$

c) $|x^2 - 6x| = x$

d) $3x - 1 = |4x^2 - x - 4|$

13. In coastal communities, the depth, d, in metres, of water in the harbour varies during the day according to the tides. The maximum depth of the water occurs at high tide and the minimum occurs at low tide. Two low tides and two high tides will generally occur over a 24-h period. On one particular day in Prince Rupert, British Columbia, the depth of the first high tide and the first low tide can be determined using the equation $|d - 4.075| = 1.665$.


a) Find the depth of the water, in metres, at the first high tide and the first low tide in Prince Rupert on this day.

b) Suppose the low tide and high tide depths for Prince Rupert on the next day are 2.94 m, 5.71 m, 2.28 m, and 4.58 m. Determine the total change in water depth that day.

14. The mass, m, in kilograms, of a bushel of wheat depends on its moisture content. Dry wheat has moisture content as low as 5% and wet wheat has moisture content as high as 50%. The equation $|m - 35.932| = 11.152$ can be used to find the extreme masses for both a dry and a wet bushel of wheat. What are these two masses?

7.4 Reciprocal Functions, pages 392–409

15. Copy each graph of $y = f(x)$ and sketch the graph of the corresponding reciprocal function, $y = \dfrac{1}{f(x)}$. Label the asymptotes, the invariant points, and the intercepts.

a)

b)
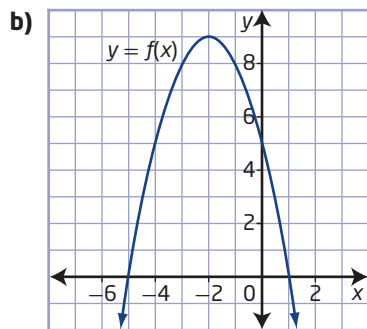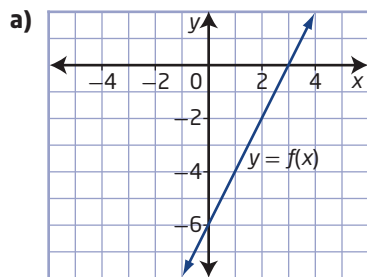
16. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes. Label the asymptotes, the invariant points, and the intercepts.

a) $f(x) = 4x - 9$ **b)** $f(x) = 2x + 5$

17. For each function,

  i) determine the corresponding reciprocal function, $y = \dfrac{1}{f(x)}$

  ii) state the non-permissible values of x and the equation(s) of the vertical asymptote(s) of the reciprocal function

  iii) determine the x-intercepts and the y-intercept of the reciprocal function

  iv) sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes

  a) $f(x) = x^2 - 25$

  b) $f(x) = x^2 - 6x + 5$

18. The force, F, in newtons (N), required to lift an object with a lever is proportional to the reciprocal of the distance, d, in metres, of the force from the fulcrum of a lever. The fulcrum is the point on which a lever pivots. Suppose this relationship can be modelled by the function $F = \dfrac{600}{d}$.
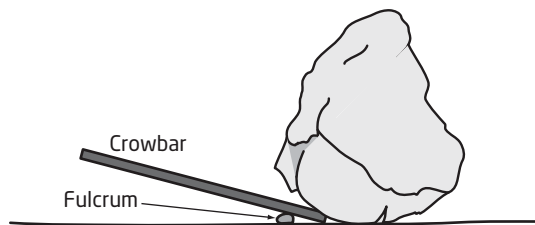

a) Determine the force required to lift an object if the force is applied 2.5 m from the fulcrum.

b) Determine the distance from the fulcrum of a 450-N force applied to lift an object.

c) How does the force needed to lift an object change if the distance from the fulcrum is doubled? tripled?

Chapter 7 Practice Test

Multiple Choice

For #1 to #5, choose the best answer.

1. The value of the expression
$|-9 - 3| - |5 - 2^3| + |-7 + 1 - 4|$ is

A 13

B 19

C 21

D 25

2. The range of the function $f(x) = |x - 3|$ is

A $\{y \mid y > 3, y \in R\}$

B $\{y \mid y \geq 3, y \in R\}$

C $\{y \mid y \geq 0, y \in R\}$

D $\{y \mid y > 0, y \in R\}$

3. The absolute value equation $|1 - 2x| = 9$ has solution(s)

A $x = -4$

B $x = 5$

C $x = -5$ and $x = 4$

D $x = -4$ and $x = 5$

4. The graph represents the reciprocal of which quadratic function?
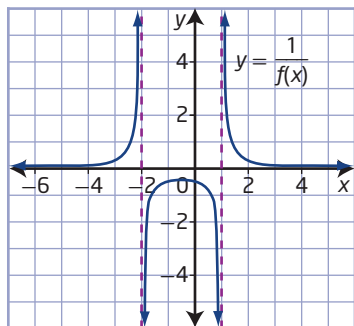
A $f(x) = x^2 + x - 2$

B $f(x) = x^2 - 3x + 2$

C $f(x) = x^2 - x - 2$

D $f(x) = x^2 + 3x + 2$

5. One of the vertical asymptotes of the graph of the reciprocal function $y = \dfrac{1}{x^2 - 16}$ has equation

A $x = 0$

B $x = 4$

C $x = 8$

D $x = 16$

Short Answer

6. Consider the function $f(x) = |2x - 7|$.

a) Sketch the graph of the function.

b) Determine the intercepts.

c) State the domain and range.

d) What is the piecewise notation form of the function?

7. Solve the equation $|3x^2 - x| = 4x - 2$ algebraically.

8. Solve the equation $|2w - 3| = w + 1$ graphically.

Extended Response

9. Determine the error(s) in the following solution. Explain how to correct the solution.

Solve $|x - 4| = x^2 + 4x$.

Case 1
$x + 4 = x^2 + 4x$
$0 = x^2 + 3x - 4$
$0 = (x + 4)(x - 1)$
$x + 4 = 0$ or $x - 1 = 0$
     $x = -4$ or $x = 1$

Case 2
$-x - 4 = x^2 + 4x$
$0 = x^2 + 5x + 4$
$0 = (x + 4)(x + 1)$
$x + 4 = 0$ or $x + 1 = 0$
   $x = -4$ or $x = -1$

The solutions are $x = -4$, $x = -1$, and $x = 1$.

10. Consider the function $f(x) = 6 - 5x$.

    a) Determine its reciprocal function.

    b) State the equations of any vertical asymptotes of the reciprocal function.

    c) Graph the function $f(x)$ and its reciprocal function. Describe a strategy that could be used to sketch the graph of any reciprocal function.

11. A biologist studying Canada geese migration analysed the vee flight formation of a particular flock using a coordinate system, in metres. The centre of each bird was assigned a coordinate point. The lead bird has the coordinates (0, 0), and the coordinates of two birds at the ends of each leg are (6.2, 15.5) and (−6.2, 15.5).

Bottom View of Flying Geese
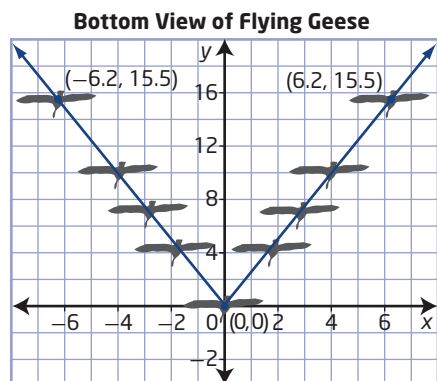
    a) Write an absolute value function whose graph contains each leg of the vee formation.

    b) What is the angle between the legs of the vee formation, to the nearest tenth of a degree?

    c) The absolute value function $y = |2.8x|$ describes the flight pattern of a different flock of geese. What is the angle between the legs of this vee formation, to the nearest tenth of a degree?

12. Astronauts in space feel lighter because weight decreases as a person moves away from the gravitational pull of Earth. Weight, W_h, in newtons (N), at a particular height, h, in kilometres, above Earth is related to the reciprocal of that height by the formula $W_h = \dfrac{W_e}{\left(\dfrac{h}{6400} + 1\right)^2}$, where W_e is the person's weight, in newtons (N), at sea level on Earth.


Canadian astronauts Julie Payette and Bob Thirsk

    a) Sketch the graph of the function for an astronaut whose weight is 750 N at sea level.

    b) Determine this astronaut's weight at a height of
      i) 8 km   **ii)** 2000 km

    c) Determine the range of heights for which this astronaut will have a weight of less than 30 N.

Did You Know?

When people go into space, their mass remains constant but their weight decreases because of the reduced gravity.

13. If a point is selected at random from a figure and is equally likely to be any point inside the figure, then the probability that a point is in the shaded region is given by

$$P = \dfrac{\text{area of shaded region}}{\text{area of entire figure}}$$

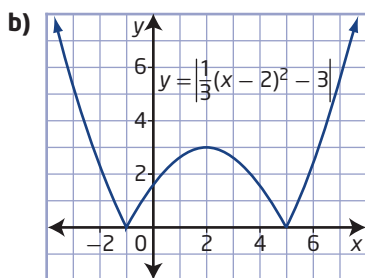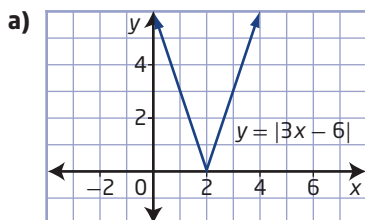What is the probability that the point is in the shaded region?
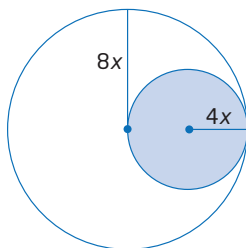
Chapter 7 Absolute Value and Reciprocal Functions

14. Order the values from least to greatest.

$|-5|, |4-6|, |2(-4)-5|, |8.4|$

15. Write the piecewise function that represents each graph.

a)

b)

16. For each absolute value function,

i) sketch the graph

ii) determine the intercepts

iii) determine the domain and range

**a)** $y = |3x - 7|$ **b)** $y = |x^2 - 3x - 4|$

17. Solve algebraically. Verify your solutions.

a) $|2x - 1| = 9$ **b)** $|2x^2 - 5| = 13$

**18.** The area, A, of a triangle on a coordinate grid with vertices at $(0, 0)$, (a, b), and (c, d) can be calculated using the formula $A = \frac{1}{2}|ad - bc|$.

a) Why do you think absolute value must be used in the formula for area?

b) Determine the area of a triangle with vertices at $(0, 0)$, $(-5, 2)$, and $(-3, 4)$.

19. Sketch the graph of $y = f(x)$ given the graph of $y = \dfrac{1}{f(x)}$. What is the original function, $y = f(x)$?
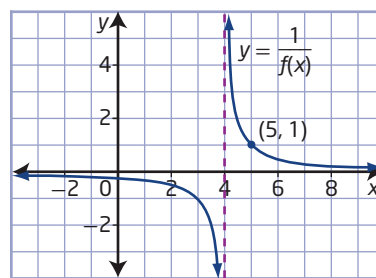
20. Copy the graph of $y = f(x)$, and sketch the graph of the reciprocal function $y = \dfrac{1}{f(x)}$. Discuss your method.
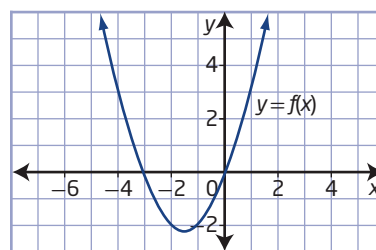
21. Sketch the graph of $y = \dfrac{1}{f(x)}$ given $f(x) = (x + 2)^2$. Label the asymptotes, the invariant points, and the intercepts.

22. Consider the function $f(x) = 3x - 1$.

a) What characteristics of the graph of $y = f(x)$ are different from those of $y = |f(x)|$?

b) Describe how the graph of $y = f(x)$ is different from the graph of $y = \dfrac{1}{f(x)}$.

Unit 3 Test

Multiple Choice

For #1 to #8, choose the best answer.

1. What is the entire radical form of $2(\sqrt[3]{-27})$?

   A $\sqrt[3]{-54}$

   B $\sqrt[3]{-108}$

   C $\sqrt[3]{-216}$

   D $\sqrt[3]{-432}$

2. What is the simplified form of $\dfrac{4\sqrt{72x^5}}{x\sqrt{8}}$, $x > 0$?

   A $\dfrac{12\sqrt{2x^3}}{\sqrt{2}}$

   B $4x\sqrt{3x}$

   C $\dfrac{6\sqrt{2x^3}}{\sqrt{2}}$

   D $12x\sqrt{x}$

3. Determine the root(s) of $x + 2 = \sqrt{x^2 + 3}$.

   A $x = -\dfrac{1}{4}$ and $x = 3$

   B $x = -\dfrac{1}{4}$

   C $x = \dfrac{1}{4}$ and $x = -3$

   D $x = \dfrac{1}{4}$

4. Simplify the rational expression $\dfrac{9x^4 - 27x^6}{3x^3}$ for all permissible values of x.

   A $3x(1 - 3x)$

   B $3x(1 - 9x^5)$

   C $3x - 9x^3$

   D $9x^3 - 9x^4$

5. Which expression could be used to determine the length of the line segment between the points $(4, -3)$ and $(-6, -3)$?

   A $-6 - 4$

   B $4 - 6$

   C $|4 - 6|$

   D $|-6 - 4|$

6. Arrange the expressions $|4 - 11|$, $\dfrac{1}{5}|-5|$, $\left|1 - \dfrac{1}{4}\right|$, and $|2| - |4|$ in order from least to greatest.

   A $|4 - 11|$, $\dfrac{1}{5}|-5|$, $\left|1 - \dfrac{1}{4}\right|$, $|2| - |4|$

   B $|2| - |4|$, $\left|1 - \dfrac{1}{4}\right|$, $\dfrac{1}{5}|-5|$, $|4 - 11|$

   C $|2| - |4|$, $\dfrac{1}{5}|-5|$, $\left|1 - \dfrac{1}{4}\right|$, $|4 - 11|$

   D $\left|1 - \dfrac{1}{4}\right|$, $\dfrac{1}{5}|-5|$, $|4 - 11|$, $|2| - |4|$

7. Which of the following statements is false?

   A $\sqrt{n}\sqrt{m} = \sqrt{mn}$

   B $\dfrac{\sqrt{18}}{\sqrt{36}} = \sqrt{\dfrac{1}{2}}$

   C $\dfrac{\sqrt{7}}{\sqrt{8n}} = \dfrac{\sqrt{14n}}{4n}$

   D $\sqrt{m^2 + n^2} = m + n$

8. The graph of $y = \dfrac{1}{f(x)}$ has vertical asymptotes at $x = -2$ and $x = 5$ and a horizontal asymptote at $y = 0$. Which of the following statements is possible?

   A $f(x) = (x + 2)(x + 5)$

   B $f(x) = x^2 - 3x - 10$

   C The domain of $f(x)$ is $\{x \mid x \neq -2, x \neq -5, x \in \mathbb{R}\}$.

   D The range of $y = \dfrac{1}{f(x)}$ is $\{y \mid y \in \mathbb{R}\}$.

Numerical Response

Copy and complete the statements in #9 to #13.

9. The radical $\sqrt{3x - 9}$ results in real numbers when $x \geq \blacksquare$.

10. When the denominator of the expression $\dfrac{\sqrt{5}}{3\sqrt{2}}$ is rationalized, the expression becomes $\blacksquare$.

11. The expression $\dfrac{3x - 7}{x + 11} - \dfrac{x - k}{x + 11}$,

$x \neq -11$, simplifies to $\dfrac{2x + 21}{x + 11}$

when the value of k is ■.

12. The lesser solution to the absolute value equation $|1 - 4x| = 9$ is $x = $ ■.

13. The graph of the reciprocal function

$f(x) = \dfrac{1}{x^2 - 4}$ has vertical asymptotes

with equations $x = $ ■ and $x = $ ■.

Written Response

14. Order the numbers from least to greatest.

$3\sqrt{7},\ 4\sqrt{5},\ 6\sqrt{2},\ 5$

15. Consider the equation $\sqrt{3x + 4} = \sqrt{2x - 5}$.

a) Describe a possible first step to solve the radical equation.

b) Determine the restrictions on the values for the variable x.

c) Algebraically determine all roots of the equation.

d) Verify the solutions by substitution.

16. Simplify the expression

$\dfrac{4x^2 + 4x - 8}{x^2 - 5x + 4} \div \dfrac{2x^2 + 3x - 2}{4x^2 + 8x - 5}$.

List all non-permissible values for the variable.

17. The diagram shows two similar triangles.
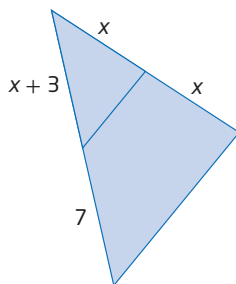
a) Write a proportion that relates the sides of the similar triangles.

b) Determine the non-permissible values for the rational equation.

c) Algebraically determine the value of x that makes the triangles similar.

18. Consider the function $y = |2x - 5|$.

a) Sketch the graph of the function.

b) Determine the intercepts.

c) State the domain and range.

d) What is the piecewise notation form of the function?

19. Solve $|x^2 - 3x| = 2$. Verify your solutions graphically.

20. Consider $f(x) = x^2 + 2x - 8$. Sketch the graph of $y = f(x)$ and the graph of

$y = \dfrac{1}{f(x)}$ on the same set of axes. Label

the asymptotes, the invariant points, and the intercepts.

21. In the sport of curling, players measure the "weight" of their shots by timing the stone between two marked lines on the ice, usually the hog lines, which are 72 ft apart. The weight, or average speed, s, of the curling stone is proportional to the reciprocal of the time, t, it takes to travel between hog lines.

Canadian women curlers at 2010 Vancouver Olympics

a) If $d = 72$, rewrite the formula $d = st$ as a function in terms of s.

b) What is the weight of a stone that takes 14.5 s to travel between hog lines?

c) How much time is required for a stone to travel between hog lines if its weight is 6.3 ft/s?