Pre-Calculus 11: Chapter 6

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CHAPTER 6

Rational Expressions and Equations

Rational expressions are used in medicine, lighting, economics, space travel, engineering, acoustics, and many other fields. For example, the rings of Saturn have puzzled astronomers since Galileo discovered them with his telescope in 1610. In October 2009, 12 years after the launch of the Cassini-Huygens project, scientists at the NASA Jet Propulsion Laboratory determined that Saturn's famous rings are neither as thin nor as flat as previously thought. Why do you think it took so long for NASA to gather and analyse this information?

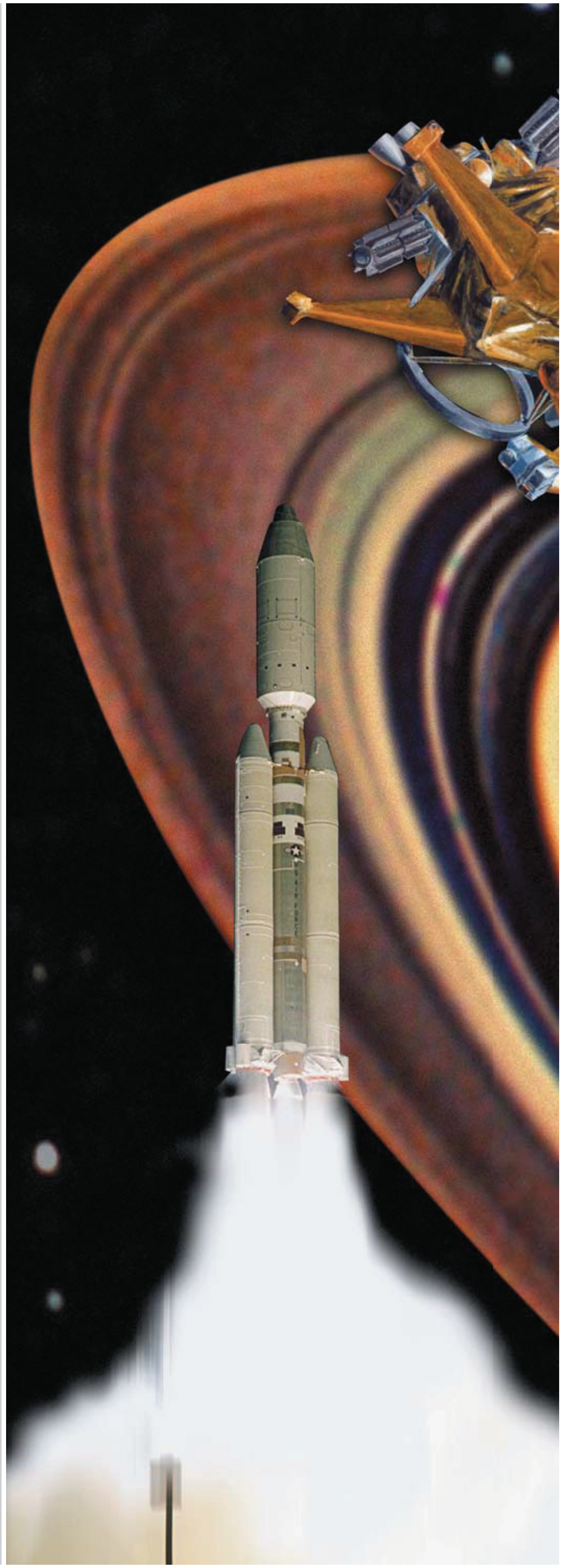
In this chapter, you will learn about the algebra of rational expressions and equations. Compare the skills you learn in the chapter with those you learned in the arithmetic of fractions. They are very similar.

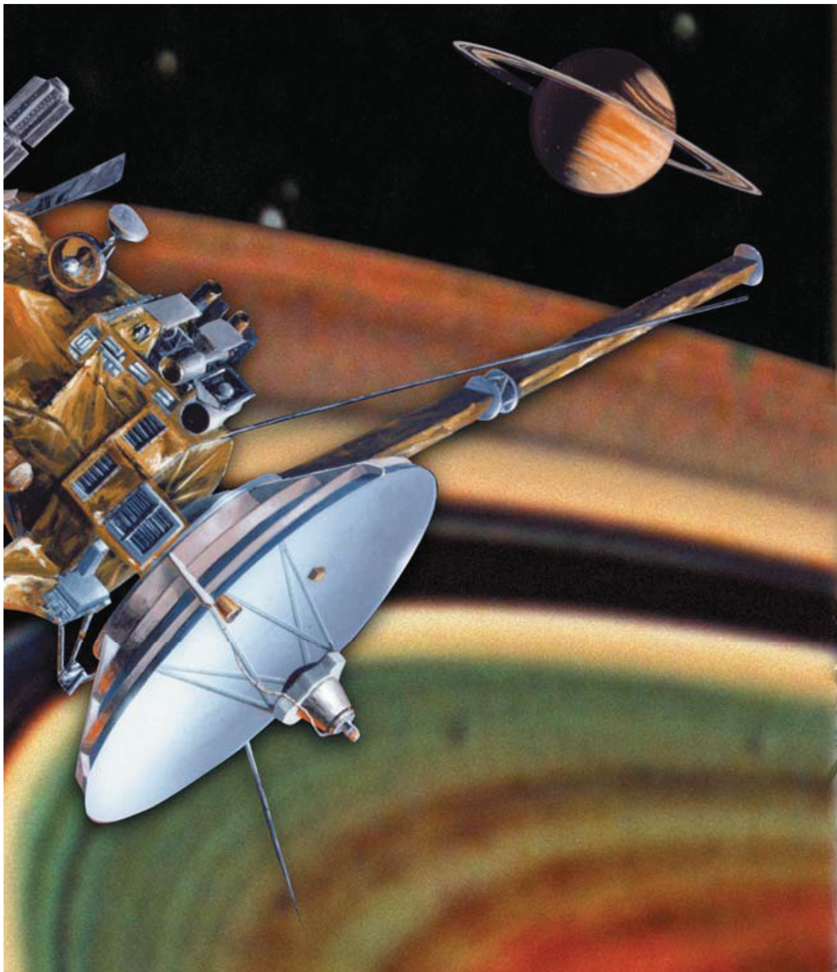
Did You Know?

The Cassini-Huygens project is a joint NASA and European Space Agency (ESA) robotic spacecraft mission to Saturn. The spacecraft was launched in 1997 and arrived to start its orbits around Saturn in 2004. The mission may continue until 2017.

Key Terms

rational expression
non-permissible value
rational equation





Career Link

You can use mathematical modelling to analyse problems in economics, science, medicine, urban planning, climate change, manufacturing, and space exploration. Building a mathematical model is normally a multi-stage process that involves writing equations, using them to predict what happens, experimenting to see if the prediction is correct, modifying the equations, and so on.

Marcel Tolkowsky, a 21-year-old Belgian mathematician, revolutionized the diamond-cutting industry in 1919 when he used mathematical modelling to calculate the formula for the ideal proportions in a cut diamond. By reducing the process to a mathematical formula, diamond-cutting is now more automated.

Web Link

To learn more about fields involving mathematical modelling, go to www.mhrprecalc11.ca and follow the links.

6.1

Rational Expressions

Focus on...

- determining non-permissible values for a rational expression
- simplifying a rational expression
- modelling a situation using a rational expression

Many day-to-day applications use **rational expressions**. You can determine the time it takes to travel across Canada by dividing the distance travelled, d, by the rate of speed at which you are travelling, r. Light intensity and the intensity of sound can be described mathematically as ratios in the form $\frac{k}{d^2}$, where k is a constant and d is the distance from the source. When would it be important to know the intensity of light or sound?

What other formulas are rational expressions?

rational expression

- an algebraic fraction with a numerator and a denominator that are polynomials
- examples are $\frac{1}{x}$, $\frac{m}{m + 1}$, $\frac{y^2 - 1}{y^2 + 2y + 1}$
- $x^2 - 4$ is a rational expression with a denominator of 1

Concert stage

Investigate Rational Expressions

Materials

- algebra tiles

1. Consider the polynomial expression $3x^2 + 12x$.

a) Use algebra tiles to model the polynomial.

b) Arrange the tiles in a rectangle to represent the length of each side as a polynomial.

c) Use the model from part b) to write a simplified form of the rational expression $\frac{3x^2 + 12x}{3x}$.

d) Are these two expressions always equivalent? Why or why not?

How can you verify that your answer is equivalent to the rational expression? Explain your reasoning.

2. Whenever you are working with algebraic fractions, it is important to determine any values that must be excluded.

a) You can write an unlimited number of arithmetic fractions, or rational numbers, of the form $\frac{a}{b}$, where a and b are integers. What integer cannot be used for b?

b) What happens in each of the following expressions when $x = 3$ is substituted?

i) $\frac{x - 7}{x - 3}$

ii) $\frac{x - 7}{x^2 - 9}$

iii) $\frac{x - 7}{x^2 - 4x + 3}$

Did You Know?

An algebraic fraction is the quotient of two algebraic expressions. Examples of algebraic fractions are $\frac{-4}{x - 2}$, $\frac{t}{5}$, 0, $\frac{x^2}{3y^5}$, and $\frac{\sqrt{3x + 1}}{x^2}$. All rational expressions are algebraic fractions but not all algebraic fractions are rational expressions.

3. What value(s) cannot be used for x in each of the following algebraic fractions?

a) $\frac{6 - x}{2x}$

b) $\frac{3}{x - 7}$

c) $\frac{4x - 1}{(x - 3)(2x + 1)}$

Reflect and Respond

4. a) What is the result when zero is divided by any non-zero number?

b) Why is division by zero undefined?

5. Write a rule that explains how to determine any values that a variable cannot be, for any algebraic fraction.

6. What operation(s) can you use when you are asked to express a rational number in lowest terms? Give examples to support your answer.

7. Describe two ways in which arithmetic fractions and algebraic fractions are similar. How do they differ?

8. a) What is the value of any rational expression in which the numerator and denominator are the same non-zero polynomials?

b) What is the value of a fraction in which the numerator and denominator are opposite integers?

c) What is the value of the rational expression $\frac{x - 3}{3 - x}$ for $x \neq 3$? Explain.

Link the Ideas

Non-Permissible Values

Whenever you use a rational expression, you must identify any values that must be excluded or are considered **non-permissible values**. Non-permissible values are all values that make the denominator zero.

non-permissible value

- any value for a variable that makes an expression undefined
- in a rational expression, a value that results in a denominator of zero
- in $\frac{x+2}{x-3}$, you must exclude the value for which $x - 3 = 0$, which is $x = 3$

Example 1

Determine Non-Permissible Values

For each rational expression, determine all non-permissible values.

a) $\frac{5t}{4sr^2}$ **b)** $\frac{3x}{x(2x-3)}$ **c)** $\frac{2p-1}{p^2-p-12}$

Solution

To determine non-permissible values, set the denominator equal to zero and solve.

a) $\frac{5t}{4sr^2}$

Determine the values for which $4sr^2 = 0$.

$4s = 0$ or $r^2 = 0$

$s = 0$ or $r = 0$

The non-permissible values are $s = 0$ and $r = 0$. The rational expression $\frac{5t}{4sr^2}$ is defined for all real numbers except $s = 0$ and $r = 0$.

This is written as $\frac{5t}{4sr^2}$, $r \neq 0$, $s \neq 0$.

b) $\frac{3x}{x(2x-3)}$

Determine the values for which $x(2x - 3) = 0$.

$x = 0$ or $2x - 3 = 0$

$x = \frac{3}{2}$

The non-permissible values are 0 and $\frac{3}{2}$.

Does it matter whether the numerator becomes zero? Explain.

c) $\frac{2p-1}{p^2-p-12}$

Determine the values for which $p^2 - p - 12 = 0$.

$(p - 4)(p + 3) = 0$ Factor $p^2 - p - 12$.

$p = 4$ or $p = -3$

The non-permissible values are 4 and -3.

Your Turn

Determine the non-permissible value(s) for each rational expression.

a) $\frac{4a}{3bc}$ **b)** $\frac{x-1}{(x+2)(x-3)}$ **c)** $\frac{2y^2}{y^2-4}$

Equivalent Rational Expressions

You can multiply or divide a rational expression by 1 and not change its value. You will create an equivalent expression using this property. For example, if you multiply $\frac{7s}{s-2}$, $s \neq 2$, by $\frac{s}{s}$, you are actually multiplying by 1, provided that $s \neq 0$.

$$\left(\frac{7s}{s-2}\right)\left(\frac{s}{s}\right) = \frac{(7s)(s)}{s(s-2)}$$
$$= \frac{7s^2}{s(s-2)},\ s \neq 0, 2$$

Why do you need to specify that $s \neq 0$?

The rational expressions $\frac{7s}{s-2}$, $s \neq 2$, and $\frac{7s^2}{s(s-2)}$, $s \neq 0, 2$, are equivalent.

Similarly, you can show that $\frac{7s}{s-2}$, $s \neq 2$, and $\frac{7s(s+2)}{(s-2)(s+2)}$, $s \neq \pm 2$, are equivalent.

What was done to the first rational expression to get the second one?

Did You Know?

Statements such as $x = 2$ and $x = -2$ can be abbreviated as $x = \pm 2$.

Simplifying Rational Expressions

Writing a rational number in lowest terms and simplifying a rational expression involve similar steps.

$$\frac{9}{12} = \frac{(\cancel{3}^{1})(3)}{(\cancel{3}_{1})(4)}$$
$$= \frac{3}{4}$$

How could you use models to determine the rational expression in simplest form?

$$\frac{m^3t}{m^2t^4} = \frac{(\cancel{m^2}^{1})(m)(\cancel{t}^{1})}{(\cancel{m^2}_{1})(\cancel{t}_{1})(t^3)}$$
$$= \frac{m}{t^3},\ m \neq 0,\ t \neq 0$$

Why is 0 a non-permissible value for the variable m in the simplified rational expression?

To simplify a rational expression, divide both the numerator and denominator by any factors that are common to the numerator and the denominator.

Recall that $\frac{AB}{AC} = \left(\frac{A}{A}\right)\left(\frac{B}{C}\right)$ and $\frac{A}{A} = 1$.

So, $\frac{AB}{AC} = \frac{B}{C}$, where A, B, and C are polynomial factors.

When a rational expression is in simplest form, or its lowest terms, the numerator and denominator have no common factors other than 1.

Example 2

Simplify a Rational Expression

Simplify each rational expression. State the non-permissible values.

a) $\dfrac{3x - 6}{2x^2 + x - 10}$

b) $\dfrac{1 - t}{t^2 - 1}$

Solution

a) $\dfrac{3x - 6}{2x^2 + x - 10}$

Factor both the numerator and the denominator. Consider the factors of the denominator to find the non-permissible values before simplifying the expression.

Why should you determine any non-permissible values before simplifying?

$$\begin{aligned}\frac{3x - 6}{2x^2 + x - 10} &= \frac{3(x - 2)}{(x - 2)(2x + 5)} \\ &= \frac{3\overset{1}{\cancel{(x - 2)}}}{\underset{1}{\cancel{(x - 2)}}(2x + 5)} \\ &= \frac{3}{2x + 5},\ x \neq 2,\ -\frac{5}{2}\end{aligned}$$

How do you obtain 2 and $-\frac{5}{2}$ as the non-permissible values?

How could you show that the initial rational expression and the simplified version are equivalent?

b) $\dfrac{1 - t}{t^2 - 1}$

Method 1: Use Factoring −1

$$\begin{aligned}\frac{1 - t}{t^2 - 1} &= \frac{1 - t}{(t - 1)(t + 1)} \\ &= \frac{-1(t - 1)}{(t - 1)(t + 1)} \\ &= \frac{-1\overset{1}{\cancel{(t - 1)}}}{\underset{1}{\cancel{(t - 1)}}(t + 1)} \\ &= \frac{-1}{t + 1},\ t \neq \pm 1\end{aligned}$$

Factor the denominator. Realize that the numerator is the opposite (additive inverse) of one of the factors in the denominator. Factor −1 from the numerator.

Method 2: Use the Property of 1

$$\begin{aligned}\frac{1-t}{t^2-1} &= \frac{1-t}{(t-1)(t+1)} \\ &= \frac{-1(1-t)}{-1(t-1)(t+1)} \\ &= \frac{\cancel{t-1}^{\,1}}{-1(\cancel{t-1})_{1}(t+1)} \\ &= \frac{1}{-1(t+1)} \\ &= \frac{-1}{t+1},\ t \neq \pm 1\end{aligned}$$

Factor the denominator. How are the numerator and the factor $(t - 1)$ in the denominator related?

Multiply the numerator and the denominator by -1.

Your Turn

Simplify each rational expression.
What are the non-permissible values?

a) $\dfrac{2y^2 + y - 10}{y^2 + 3y - 10}$

b) $\dfrac{6 - 2m}{m^2 - 9}$

Example 3

Rational Expressions With Pairs of Non-Permissible Values

Consider the expression $\dfrac{16x^2 - 9y^2}{8x - 6y}$.

a) What expression represents the non-permissible values for x?
b) Simplify the rational expression.
c) Evaluate the expression for $x = 2.6$ and $y = 1.2$.
Show two ways to determine the answer.

Solution

a) $\dfrac{16x^2 - 9y^2}{8x - 6y}$

Determine an expression for x for which $8x - 6y = 0$.

$x = \dfrac{6y}{8}$ or $\dfrac{3y}{4}$

x cannot have a value of $\dfrac{3y}{4}$ or the denominator will be zero and the expression will be undefined. The expression for the non-permissible values of x is $x = \dfrac{3y}{4}$.

What is an expression for the non-permissible values of y?

Examples of non-permissible values include $\left(\dfrac{3}{4}, 1\right)$, $\left(\dfrac{3}{2}, 2\right)$, $\left(\dfrac{9}{4}, 3\right)$, and so on.

b) $\dfrac{16x^2 - 9y^2}{8x - 6y} = \dfrac{(4x - 3y)(4x + 3y)}{2(4x - 3y)}$

$= \dfrac{\overset{1}{\cancel{(4x - 3y)}}(4x + 3y)}{2\underset{1}{\cancel{(4x - 3y)}}}$

$= \dfrac{4x + 3y}{2}, x \neq \dfrac{3y}{4}$

Factor the numerator and the denominator.

What are you assuming when you divide both the numerator and the denominator by $4x - 3y$?

c) First, check that the values $x = 2.6$ and $y = 1.2$ are permissible.

Left Side	Right Side
x	$\dfrac{3y}{4}$
$= 2.6$	$= \dfrac{3(1.2)}{4}$
	$= 0.9$

Left Side $\neq$ Right Side

Thus, $x \neq \dfrac{3y}{4}$ for $x = 2.6$ and $y = 1.2$, so the values are permissible.

Method 1: Substitute Into the Original Rational Expression

$\dfrac{16x^2 - 9y^2}{8x - 6y} = \dfrac{16(2.6)^2 - 9(1.2)^2}{8(2.6) - 6(1.2)}$

$= \dfrac{95.2}{13.6}$

$= 7$

Method 2: Substitute Into the Simplified Rational Expression

$\dfrac{4x + 3y}{2} = \dfrac{4(2.6) + 3(1.2)}{2}$

$= \dfrac{14}{2}$

$= 7$

The value of the expression when $x = 2.6$ and $y = 1.2$ is 7.

Your Turn

Use the rational expression $\dfrac{16x^2 - 9y^2}{8x - 6y}$ to help answer the following.

a) What is the non-permissible value for y if $x = 3$?

b) Evaluate the expression for $x = 1.5$ and $y = 2.8$.

c) Give a reason why it may be beneficial to simplify a rational expression.

Key Ideas

- A rational expression is an algebraic fraction of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.
- A non-permissible value is a value of the variable that causes an expression to be undefined. For a rational expression, this occurs when the denominator is zero.
- Rational expressions can be simplified by:
  - factoring the numerator and the denominator
  - determining non-permissible values
  - dividing both the numerator and denominator by all common factors

Check Your Understanding

Practise

1. What should replace ■ to make the expressions in each pair equivalent?

a) $\frac{3}{5}, \frac{■}{30}$

b) $\frac{2}{5}, \frac{■}{35x}, x \neq 0$

c) $\frac{4}{■}, \frac{44}{77}$

d) $\frac{x + 2}{x - 3}, \frac{4x + 8}{■}, x \neq 3$

e) $\frac{3(6)}{■(6)}, \frac{3}{8}$

f) $\frac{1}{y - 2}, \frac{■}{y^2 - 4}, y \neq \pm 2$

2. State the operation and quantity that must be applied to both the numerator and the denominator of the first expression to obtain the second expression.

a) $\frac{3p^2q}{pq^2}, \frac{3p}{q}$

b) $\frac{2}{x + 4}, \frac{2x - 8}{x^2 - 16}$

c) $\frac{-4(m - 3)}{m^2 - 9}, \frac{-4}{m + 3}$

d) $\frac{1}{y - 1}, \frac{y^2 + y}{y^3 - y}$

3. What value(s) of the variable, if any, make the denominator of each expression equal zero?

a) $\frac{-4}{x}$

b) $\frac{3c - 1}{c - 1}$

c) $\frac{y}{y + 5}$

d) $\frac{m + 3}{5}$

e) $\frac{1}{d^2 - 1}$

f) $\frac{x - 1}{x^2 + 1}$

4. Determine the non-permissible value(s) for each rational expression. Why are these values not permitted?

a) $\frac{3a}{4 - a}$

b) $\frac{2e + 8}{e}$

c) $\frac{3(y + 7)}{(y - 4)(y + 2)}$

d) $\frac{-7(r - 1)}{(r - 1)(r + 3)}$

e) $\frac{2k + 8}{k^2}$

f) $\frac{6x - 8}{(3x - 4)(2x + 5)}$

5. What value(s) for the variables must be excluded when working with each rational expression?

a) $\frac{4\pi r^2}{8\pi r^3}$

b) $\frac{2t + t^2}{t^2 - 1}$

c) $\frac{x - 2}{10 - 5x}$

d) $\frac{3g}{g^3 - 9g}$

6. Simplify each rational expression. State any non-permissible values for the variables.

a) $\frac{2c(c - 5)}{3c(c - 5)}$

b) $\frac{3w(2w + 3)}{2w(3w + 2)}$

c) $\frac{(x - 7)(x + 7)}{(2x - 1)(x - 7)}$

d) $\frac{5(a - 3)(a + 2)}{10(3 - a)(a + 2)}$

7. Consider the rational expression $\frac{x^2 - 1}{x^2 + 2x - 3}$.

a) Explain why you cannot divide out the x^2 in the numerator with the x^2 in the denominator.

b) Explain how to determine the non-permissible values. State the non-permissible values.

c) Explain how to simplify a rational expression. Simplify the rational expression.

8. Write each rational expression in simplest form. State any non-permissible values for the variables.

a) $\frac{6r^2p^3}{4rp^4}$

b) $\frac{3x - 6}{10 - 5x}$

c) $\frac{b^2 + 2b - 24}{2b^2 - 72}$

d) $\frac{10k^2 + 55k + 75}{20k^2 - 10k - 150}$

e) $\frac{x - 4}{4 - x}$

f) $\frac{5(x^2 - y^2)}{x^2 - 2xy + y^2}$

Apply

9. Since $\frac{x^2 + 2x - 15}{x - 3}$ can be written as $\frac{(x - 3)(x + 5)}{x - 3}$, you can say that $\frac{x^2 + 2x - 15}{x - 3}$ and $x + 5$ are equivalent expressions. Is this statement always, sometimes, or never true? Explain.

10. Explain why 6 may not be the only non-permissible value for a rational expression that is written in simplest form as $\frac{y}{y - 6}$. Give examples to support your answer.

11. Mike always looks for shortcuts. He claims, "It is easy to simplify expressions such as $\frac{5 - x}{x - 5}$ because the top and bottom are opposites of each other and any time you divide opposites the result is –1." Is Mike correct? Explain why or why not.

12. Suppose you are tutoring a friend in simplifying rational expressions. Create three sample expressions written in the form $\frac{ax^2 + bx + c}{dx^2 + ex + f}$ where the numerators and denominators factor and the expressions can be simplified. Describe the process you used to create one of your expressions.

13. Shali incorrectly simplifies a rational expression as shown below.

$$\begin{aligned}\frac{g^2 - 4}{2g - 4} &= \frac{(g - 2)(g + 2)}{2(g - 2)}\\ &= \frac{g + 2}{2}\\ &= g + 1\end{aligned}$$

What is Shali's error? Explain why the step is incorrect. Show the correct solution.

14. Create a rational expression with variable p that has non-permissible values of 1 and -2.

15. The distance, d, can be determined using the formula $d = rt$, where r is the rate of speed and t is the time.

a) If the distance is represented by $2n^2 + 11n + 12$ and the rate of speed is represented by $2n^2 - 32$, what is an expression for the time?

b) Write your expression from part a) in simplest form. Identify any non-permissible values.

16. You have been asked to draw the largest possible circle on a square piece of paper. The side length of the piece of paper is represented by $2x$.

a) Draw a diagram showing your circle on the piece of paper. Label your diagram.

b) Create a rational expression comparing the area of your circle to the area of the piece of paper.

c) Identify any non-permissible values for your rational expression.

d) What is your rational expression in simplest form?

e) What percent of the paper is included in your circle? Give your answer to the nearest percent.

17. A chemical company is researching the effect of a new pesticide on crop yields. Preliminary results show that the extra yield per hectare is given by the expression $\frac{900p}{2 + p}$, where p is the mass of pesticide, in kilograms. The extra yield is also measured in kilograms.

a) Explain whether the non-permissible value needs to be considered in this situation.

b) What integral value for p gives the least extra yield?

c) Substitute several values for p and determine what seems to be the greatest extra yield possible.

18. Write an expression in simplest form for the time required to travel 100 km at each rate. Identify any non-permissible values.

a) $2q$ kilometres per hour

b) $(p - 4)$ kilometres per hour

19. A school art class is planning a day trip to the Glenbow Museum in Calgary. The cost of the bus is \$350 and admission is \$9 per student.

a) What is the total cost for a class of 30 students?

b) Write a rational expression that could be used to determine the cost per student if n students go on the trip.

c) Use your expression to determine the cost per student if 30 students go.

Did You Know?

The Glenbow Museum in Calgary is one of western Canada's largest museums. It documents life in western Canada from the 1800s to the present day. Exhibits trace the traditions of the First Nations peoples as well as the hardships of ranching and farming in southern Alberta.

First Nations exhibit at Glenbow Museum

20. Terri believes that $\frac{5}{m + 5}$ can be expressed in simplest form as $\frac{1}{m + 1}$.

a) Do you agree with Terri? Explain in words.

b) Use substitution to show whether $\frac{5}{m + 5}$ and $\frac{1}{m + 1}$ are equivalent or not.

21. Sometimes it is useful to write more complicated equivalent rational expressions. For example, $\frac{3x}{4}$ is equivalent to $\frac{15x}{20}$ and to $\frac{3x^2 - 6x}{4x - 8}$, $x \neq 2$.

a) How can you change $\frac{3x}{4}$ into its equivalent form, $\frac{15x}{20}$?

b) What do you need to do to $\frac{3x}{4}$ to get $\frac{3x^2 - 6x}{4x - 8}$?

22. Write a rational expression equivalent to $\frac{x - 2}{3}$ that has

a) a denominator of 12

b) a numerator of $3x - 6$

c) a denominator of $6x + 15$

23. Write a rational expression that satisfies each set of conditions.

a) equivalent to 5, with $5b$ as the denominator

b) equivalent to $\frac{x + 1}{3}$, with a denominator of $12a^2b$

c) equivalent to $\frac{a - b}{7x}$, with a numerator of $2b - 2a$

24. The area of right △PQR is $(x^2 - x - 6)$ square units, and the length of side PQ is $(x - 3)$ units. Side PR is the hypotenuse.

a) Draw a diagram of △PQR.

b) Write an expression for the length of side QR. Express your answer in simplest form.

c) What are the non-permissible values?

25. The work shown to simplify each rational expression contains at least one error. Rewrite each solution, correcting the errors.

a) $$\begin{aligned}\frac{6x^2 - x - 1}{9x^2 - 1} &= \frac{(2x + 1)(3x - 1)}{(3x + 1)(3x - 1)} \\ &= \frac{2x + 1}{3x + 1},\ x \neq -\frac{1}{3}\end{aligned}$$

b) $$\begin{aligned}\frac{2n^2 + n - 15}{5n - 2n^2} &= \frac{(n + 3)(2n - 5)}{n(5 - 2n)} \\ &= \frac{(n + 3)(2n - 5)}{-n(2n - 5)} \\ &= \frac{n + 3}{-n} \\ &= \frac{n - 3}{n},\ n \neq 0, \frac{5}{2}\end{aligned}$$

Extend

26. Write in simplest form. Identify any non-permissible values.

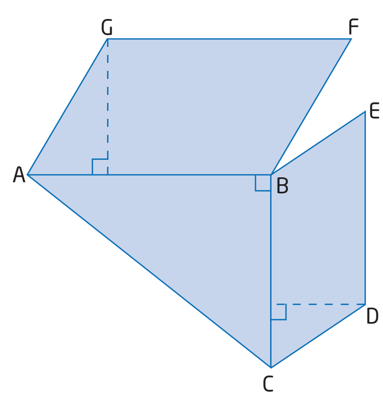
a) $\frac{(x + 2)^2 - (x + 2) - 20}{x^2 - 9}$

b) $\frac{4(x^2 - 9)^2 - (x + 3)^2}{x^2 + 6x + 9}$

c) $\frac{(x^2 - x)^2 - 8(x^2 - x) + 12}{(x^2 - 4)^2 - (x - 2)^2}$

d) $\frac{(x^2 + 4x + 4)^2 - 10(x^2 + 4x + 4) + 9}{(2x + 1)^2 - (x + 2)^2}$

27. Parallelogram ABFG has an area of $(16x^2 - 1)$ square units and a height of $(4x - 1)$ units. Parallelogram BCDE has an area of $(6x^2 - x - 12)$ square units and a height of $(2x - 3)$ units. What is an expression for the area of △ABC? Leave your answer in the form $ax^2 + bx + c$. What are the non-permissible values?



28. Carpet sellers need to know if a partial roll contains enough carpet to complete an order. Your task is to create an expression that gives the approximate length of carpet on a roll using measurements from the end of the roll.

a) Let t represent the thickness of the carpet and L the length of carpet on a roll. What is an expression for the area of the rolled edge of the carpet?

b) Draw a diagram showing the carpet rolled on a centre tube. Label the radius of the centre tube as r and the radius of the entire carpet roll as R. What is an expression for the approximate area of the edge of the carpet on the roll? Write your answer in factored form.

c) Write an expression for the length of carpet on a roll of thickness t. What conditions apply to t, L, R, and r?

Create Connections

29. Write a rational expression using one variable that satisfies the following conditions.

a) The non-permissible values are -2 and 5.

b) The non-permissible values are 1 and -3 and the expression is $\frac{x}{x - 1}$ in simplest form. Explain how you found your expression.

30. Consider the rational expressions $\frac{y - 3}{4}$ and $\frac{2y^2 - 5y - 3}{8y + 4}$, $y \neq -\frac{1}{2}$.

a) Substitute a value for y to show that the two expressions are equivalent.

b) Use algebra to show that the expressions are equivalent.

c) Which approach proves the two expressions are equivalent? Why?

31. Two points on a coordinate grid are represented by A(p, 3) and B($2p + 1$, $p - 5$).

a) Write a rational expression for the slope of the line passing through A and B. Write your answer in simplest form.

b) Determine a value for p such that the line passing through A and B has a negative slope.

c) Describe the line through A and B for any non-permissible value of p.

32. Use examples to show how writing a fraction in lowest terms and simplifying a rational expression involve the same mathematical processes.

6.2

Multiplying and Dividing Rational Expressions

Focus on…

- comparing operations on rational expressions to the same operations on rational numbers
- identifying non-permissible values when performing operations on rational expressions
- determining the product or quotient of rational expressions in simplest form

Aboriginal House, University of Manitoba

Aboriginal House at the University of Manitoba earned gold LEED status in December 2009 for combining aboriginal values and international environmental practices. According to Elder Garry Robson, the cultural significance of the building has many layers. Elder Robson says, "Once you've learned one (layer), then you learn another and another and so on."

How does Elder Robson's observation apply in mathematics?

Did You Know?

The Leadership in Energy and Environmental Design (LEED) rating system is a nationally accepted standard for constructing and operating green buildings. It promotes sustainability in site development, water and energy efficiency, materials selection, and indoor environmental quality.

Investigate Multiplying and Dividing Rational Expressions

Materials

- grid paper

1. Determine the product $\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)$. Describe a pattern that could be used to numerically determine the answer.

2. Multiplying rational expressions follows the same pattern as multiplying rational numbers. Use your pattern from step 1 to help you multiply $\frac{x + 3}{2}$ and $\frac{x + 1}{4}$.

3. Determine the value of $\frac{2}{3} \div \frac{1}{6}$. Describe a pattern that could be used as a method to divide rational numbers.

4. Apply your method from step 3 to express $\frac{x-3}{x^2-9} \div \frac{x}{x+3}$ in simplest form. Do you think your method always works? Why?
5. What are the non-permissible values for x in step 4? Explain how to determine the non-permissible values.

Reflect and Respond

6. Explain how you can apply the statement "once you've learned one, then you learn another and another" to the mathematics of rational numbers and rational expressions.
7. Describe a process you could follow to find the product in simplest form when multiplying rational expressions.
8. Describe a process for dividing rational expressions and expressing the answer in simplest form. Show how your process works using an example.
9. Explain why it is important to identify all non-permissible values before simplifying when using rational expressions.

Link the Ideas

Multiplying Rational Expressions

When you multiply rational expressions, you follow procedures similar to those for multiplying rational numbers.

$$\begin{aligned}\left(\frac{5}{8}\right)\left(\frac{4}{15}\right) &= \frac{(5)(4)}{(8)(15)}\\ &= \frac{(5)(4)}{(2)(4)(3)(5)}\\ &= \frac{\overset{1}{\cancel{(5)}}\overset{1}{\cancel{(4)}}}{2\underset{1}{\cancel{(4)}}(3)\underset{1}{\cancel{(5)}}}\\ &= \frac{1}{6}\end{aligned}$$

$$\begin{aligned}\left(\frac{4x^2}{3xy}\right)\left(\frac{y^2}{8x}\right) &= \frac{(4x^2)(y^2)}{(3xy)(8x)}\\ &= \frac{\overset{1}{\cancel{4}}\overset{1}{\cancel{x^2}}\overset{y}{\cancel{y^2}}}{\underset{6}{\cancel{24}}\underset{1}{\cancel{x^2}}\underset{1}{\cancel{y}}}\\ &= \frac{y}{6}, x \neq 0, y \neq 0\end{aligned}$$

Values for the variables that result in any denominator of zero are non-permissible. Division by zero is not defined in the real-number system.

Example 1

Multiply Rational Expressions

Multiply. Write your answer in simplest form.
Identify all non-permissible values.

$$\frac{a^2 - a - 12}{a^2 - 9} \times \frac{a^2 - 4a + 3}{a^2 - 4a}$$

Solution

Factor each numerator and denominator.

$$\begin{aligned}
\frac{a^2 - a - 12}{a^2 - 9} \times \frac{a^2 - 4a + 3}{a^2 - 4a} &= \frac{(a - 4)(a + 3)}{(a - 3)(a + 3)} \times \frac{(a - 3)(a - 1)}{a(a - 4)} \\
&= \frac{(a - 4)(a + 3)(a - 3)(a - 1)}{(a - 3)(a + 3)(a)(a - 4)} \\
&= \frac{\cancelto{1}{(a - 4)}\cancelto{1}{(a + 3)}\cancelto{1}{(a - 3)}(a - 1)}{\cancelto{1}{(a - 3)}\cancelto{1}{(a + 3)}(a)\cancelto{1}{(a - 4)}} \\
&= \frac{a - 1}{a}
\end{aligned}$$

The non-permissible values are $a = -3, 0, 3$, and 4, since these values give zero in the denominator of at least one fraction, and division by zero is not permitted in the real numbers.

$$\frac{a^2 - a - 12}{a^2 - 9} \times \frac{a^2 - 4a + 3}{a^2 - 4a} = \frac{a - 1}{a},\ a \neq -3, 0, 3, 4$$

Where is the best place to look when identifying non-permissible values in products of rational expressions?

Your Turn

Express each product in simplest form.
What are the non-permissible values?

a) $\dfrac{d}{2\pi r} \times \dfrac{2\pi rh}{d - 2}$

b) $\dfrac{y^2 - 9}{r^3 - r} \times \dfrac{r^2 - r}{y + 3}$

Dividing Rational Expressions

Dividing rational expressions follows similar procedures to those for dividing rational numbers.

Method 1: Use a Common Denominator

$$\frac{5}{3} \div \frac{1}{6} = \frac{10}{6} \div \frac{1}{6} \qquad\qquad \frac{3x^2}{y^2} \div \frac{x}{y} = \frac{3x^2}{y^2} \div \frac{xy}{y^2}$$

$$= \frac{10}{1} \qquad\qquad = \frac{3x^2}{xy}$$

$$= 10 \qquad\qquad = \frac{3x}{y},\ x \neq 0,\ y \neq 0$$

Why is $x = 0$ a non-permissible value?

Method 2: Multiply by the Reciprocal

$$\frac{5}{3} \div \frac{1}{6} = \frac{5}{3} \times \frac{6}{1} \qquad\qquad \frac{3x^2}{y^2} \div \frac{x}{y} = \frac{3x^2}{y^2} \times \frac{y}{x}$$

$$= 10 \qquad\qquad = \frac{3x}{y},\ x \neq 0,\ y \neq 0$$

Example 2

Divide Rational Expressions

Determine the quotient in simplest form. Identify all non-permissible values.

$$\frac{x^2 - 4}{x^2 - 4x} \div \frac{x^2 + x - 6}{x^2 + x - 20}$$

Solution

$$\frac{x^2 - 4}{x^2 - 4x} \div \frac{x^2 + x - 6}{x^2 + x - 20}$$

$$= \frac{(x + 2)(x - 2)}{x(x - 4)} \div \frac{(x + 3)(x - 2)}{(x + 5)(x - 4)}$$

Factor.

$$= \frac{(x + 2)(x - 2)}{x(x - 4)} \times \frac{(x + 5)(x - 4)}{(x + 3)(x - 2)}$$

Use similar procedures for dividing rational expressions as for dividing fractions. Recall that dividing by a fraction is the same as multiplying by its reciprocal.

$$= \frac{(x + 2)\overset{1}{\cancel{(x - 2)}}(x + 5)\overset{1}{\cancel{(x - 4)}}}{x\underset{1}{\cancel{(x - 4)}}(x + 3)\underset{1}{\cancel{(x - 2)}}}$$

$$= \frac{(x + 2)(x + 5)}{x(x + 3)},\ x \neq -5, -3, 0, 2, 4$$

What was done to get this simpler answer?

The non-permissible values for x are -5, -3, 0, 2, and 4.

Which step(s) should you look at to determine non-permissible values?

Your Turn

Simplify. What are the non-permissible values?

$$\frac{c^2 - 6c - 7}{c^2 - 49} \div \frac{c^2 + 8c + 7}{c^2 + 7c}$$

Did You Know?

A complex rational expression contains a fraction in both the numerator and denominator. The expression in Example 2 could also be written as the complex rational expression

$$\frac{\dfrac{x^2 - 4}{x^2 - 4x}}{\dfrac{x^2 + x - 6}{x^2 + x - 20}}$$

Example 3

Multiply and Divide Rational Expressions

Simplify. What are the non-permissible values?

$$\frac{2m^2 - 7m - 15}{2m^2 - 10m} \div \frac{4m^2 - 9}{6} \times (3 - 2m)$$

Solution

$$\frac{2m^2 - 7m - 15}{2m^2 - 10m} \div \frac{4m^2 - 9}{6} \times (3 - 2m)$$

$$= \frac{(2m + 3)(m - 5)}{2m(m - 5)} \div \frac{(2m - 3)(2m + 3)}{6} \times (3 - 2m)$$

Apply the order of operations.

$$= \frac{(2m + 3)(m - 5)}{2m(m - 5)} \times \frac{6}{(2m - 3)(2m + 3)} \times \frac{-1(2m - 3)}{1}$$

How do you know that $3 - 2m$ and $-1(2m - 3)$ are equivalent?

$$= \frac{\cancel{(2m + 3)}^{1}\cancel{(m - 5)}^{1}(\cancel{6}^{3})(-1)\cancel{(2m - 3)}^{1}}{\cancel{2}_{1}m\cancel{(m - 5)}_{1}\cancel{(2m - 3)}_{1}\cancel{(2m + 3)}_{1}}$$

$$= -\frac{3}{m},\ m \neq -\frac{3}{2},\ 0,\ 5,\ \frac{3}{2}$$

Where do these non-permissible values come from?

The non-permissible values for m are $\pm\frac{3}{2}$, 0, and 5.

Your Turn

Simplify. Identify all non-permissible values.

$$\frac{3x + 12}{3x^2 - 5x - 12} \div \frac{12}{3x + 4} \times \frac{2x - 6}{x + 4}$$

Key Ideas

- Multiplying rational expressions is similar to multiplying rational numbers. Factor each numerator and denominator. Identify any non-permissible values. Divide both the numerator and the denominator by any common factors to create a simplified expression.

$$\frac{2}{3} \times \frac{9}{8} = \frac{2}{3} \times \frac{(3)(3)}{2(4)} = \frac{\cancel{(2)}^{1}\cancel{(3)}^{1}(3)}{\cancel{(3)}_{1}\cancel{(2)}_{1}(4)} = \frac{3}{4}$$

$$\frac{2}{b - 3} \times \frac{b^2 - 9}{4b} = \frac{2}{b - 3} \times \frac{(b - 3)(b + 3)}{4b} = \frac{\cancel{2}^{1}\cancel{(b - 3)}^{1}(b + 3)}{\cancel{4}_{2}b\cancel{(b - 3)}_{1}} = \frac{b + 3}{2b},\ b \neq 0,\ 3$$

- Dividing rational expressions is similar to dividing fractions. Convert division to multiplication by multiplying by the reciprocal of the divisor.

$$\frac{2}{3} \div \frac{4}{9} = \frac{2}{3} \times \frac{9}{4} \qquad \frac{2(x - 1)}{3} \div \frac{(x - 1)(x + 1)}{5} = \frac{2(x - 1)}{3} \times \frac{5}{(x - 1)(x + 1)}$$

- When dividing, no denominator can equal zero. In $\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C}$, the non-permissible values are $B = 0$, $C = 0$, and $D = 0$.

Check Your Understanding

Practise

1. Simplify each product. Identify all non-permissible values.
   a) $\dfrac{12m^2f}{5cf} \times \dfrac{15c}{4m}$
   b) $\dfrac{3(a - b)}{(a - 1)(a + 5)} \times \dfrac{(a - 5)(a + 5)}{15(a - b)}$
   c) $\dfrac{(y - 7)(y + 3)}{(2y - 3)(2y + 3)} \times \dfrac{4(2y + 3)}{(y + 3)(y - 1)}$

2. Write each product in simplest form. Determine all non-permissible values.
   a) $\dfrac{d^2 - 100}{144} \times \dfrac{36}{d + 10}$
   b) $\dfrac{a + 3}{a + 1} \times \dfrac{a^2 - 1}{a^2 - 9}$
   c) $\dfrac{4z^2 - 25}{2z^2 - 13z + 20} \times \dfrac{z - 4}{4z + 10}$
   d) $\dfrac{2p^2 + 5p - 3}{2p - 3} \times \dfrac{p^2 - 1}{6p - 3} \times \dfrac{2p - 3}{p^2 + 2p - 3}$

3. What is the reciprocal of each rational expression?
   a) $\dfrac{2}{t}$
   b) $\dfrac{2x - 1}{3}$
   c) $\dfrac{-8}{3 - y}$
   d) $\dfrac{2p - 3}{p - 3}$

4. What are the non-permissible values in each quotient?
   a) $\dfrac{4t^2}{3s} \div \dfrac{2t}{s^2}$
   b) $\dfrac{r^2 - 7r}{r^2 - 49} \div \dfrac{3r^2}{r + 7}$
   c) $\dfrac{5}{n + 1} \div \dfrac{10}{n^2 - 1} \div (n - 1)$

5. What is the simplified product of $\dfrac{2x - 6}{x + 3}$ and $\dfrac{x + 3}{2}$? Identify any non-permissible values.

6. What is the simplified quotient of $\dfrac{y^2}{y^2 - 9}$ and $\dfrac{y}{y - 3}$? Identify any non-permissible values.

7. Show how to simplify each rational expression or product.
   a) $\dfrac{3 - p}{p - 3}$
   b) $\dfrac{7k - 1}{3k} \times \dfrac{1}{1 - 7k}$

8. Express each quotient in simplest form. Identify all non-permissible values.
   a) $\dfrac{2w^2 - w - 6}{3w + 6} \div \dfrac{2w + 3}{w + 2}$
   b) $\dfrac{v - 5}{v} \div \dfrac{v^2 - 2v - 15}{v^3}$
   c) $\dfrac{9x^2 - 1}{x + 5} \div \dfrac{3x^2 - 5x - 2}{2 - x}$
   d) $\dfrac{8y^2 - 2y - 3}{y^2 - 1} \div \dfrac{2y^2 - 3y - 2}{2y - 2} \div \dfrac{3 - 4y}{y + 1}$

9. Explain why the non-permissible values in the quotient $\dfrac{x - 5}{x + 3} \div \dfrac{x + 1}{x - 2}$ are -3, -1 and 2.

Apply

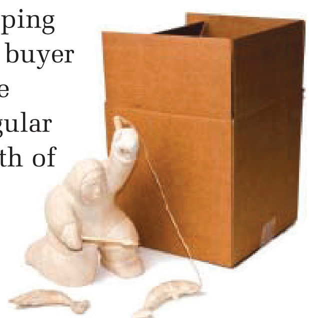
10. The height of a stack of plywood is represented by $\dfrac{n^2 - 4}{n + 1}$. If the number of sheets is defined by $n - 2$, what expression could be used to represent the thickness of one sheet? Express your answer in simplest form.

11. Write an expression involving a product or a quotient of rational expressions for each situation. Simplify each expression.

a) The Mennonite Heritage Village in Steinbach, Manitoba, has a working windmill. If the outer end of a windmill blade turns at a rate of $\frac{x - 3}{5}$ metres per minute, how far does it travel in 1 h?

b) A plane travels from Victoria to Edmonton, a distance of 900 km, in $\frac{600}{n + 1}$ hours. What is the average speed of the plane?

c) Simione is shipping his carving to a buyer in Winnipeg. He makes a rectangular box with a length of $(2x - 3)$ metres and a width of $(x + 1)$ metres. The volume of the box is $(x^2 + 2x + 1)$ cubic metres. What is an expression for the height of the box?



12. How does the quotient of $\frac{3m + 1}{m - 1}$ and $\frac{3m + 1}{m^2 - 1}$ compare to the quotient of $\frac{3m + 1}{m^2 - 1}$ and $\frac{3m + 1}{m - 1}$? Is this always true or sometimes true? Explain your thinking.

13. Simplifying a rational expression is similar to using unit analysis to convert from one unit to another. For example, to convert 68 cm to kilometres, you can use the following steps.

$$(68\text{ cm})\left(\frac{1\text{ m}}{100\text{ cm}}\right)\left(\frac{1\text{ km}}{1000\text{ m}}\right)$$
$$= (68\ \cancel{\text{cm}}) \times \left(\frac{1\ \cancel{\text{m}}}{100\ \cancel{\text{cm}}}\right) \times \left(\frac{1\text{ km}}{1000\ \cancel{\text{m}}}\right)$$
$$= \frac{68\text{ km}}{(100)(1000)}$$
$$= 0.000\ 68\text{ km}$$

Therefore, 68 cm is equivalent to 0.000 68 km. Create similar ratios that you can use to convert a measurement in yards to its equivalent in centimetres. Use 1 in. = 2.54 cm. Provide a specific example.

14. Tessa is practising for a quiz. Her work on one question is shown below.

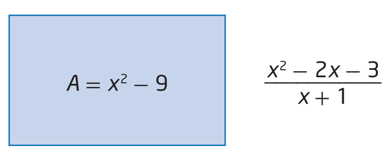
$$\frac{c^2 - 36}{2c} \div \frac{c + 6}{8c^2}$$
$$= \frac{2c}{(c - 6)(c + 6)} \times \frac{c + 6}{(2c)(4c)}$$
$$= \frac{\overset{1}{\cancel{2c}}}{(c - 6)\underset{1}{\cancel{(c + 6)}}} \times \frac{\overset{1}{\cancel{c + 6}}}{\underset{1}{\cancel{(2c)}}(4c)}$$
$$= \frac{1}{4c(c - 6)}$$

a) Identify any errors that Tessa made.

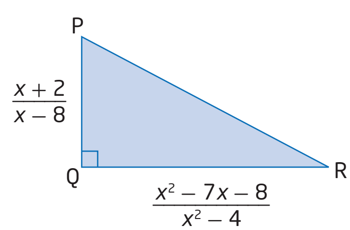
b) Complete the question correctly.

c) How does the correct answer compare with Tessa's answer? Explain.

15. Write an expression to represent the length of the rectangle. Simplify your answer.



16. What is an expression for the area of △PQR? Give your answer in simplest form.



17. You can manipulate variables in a formula by substituting from one formula into another. Try this on the following. Give all answers in simplest form.

a) If $K = \frac{P}{2m}$ and $m = \frac{h}{w}$, express K in terms of P, h, and w.

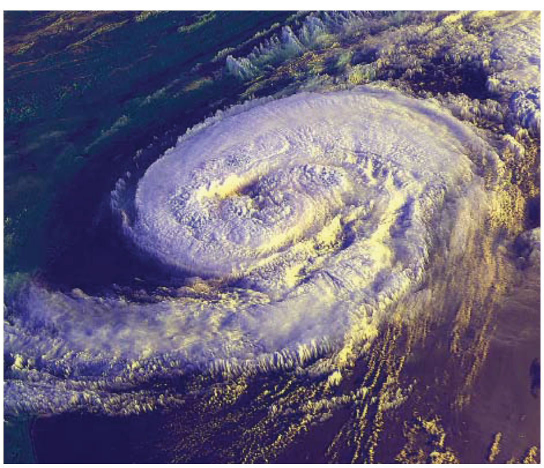
b) If $y = \frac{2\pi}{d}$ and $x = dr$, express y in terms of π, r, and x.

c) Use the formulas $v = wr$ and $a = w^2r$ to determine a formula for a in terms of v and w.

18. The volume, V, of a gas increases or decreases with its temperature, T, according to Charles's law by the formula $\frac{V_1}{V_2} = \frac{T_1}{T_2}$. Determine V_1 if $V_2 = \frac{n^2 - 16}{n - 1}$, $T_1 = \frac{n - 1}{3}$, and $T_2 = \frac{n + 4}{6}$.
Express your answer in simplest form.

Did You Know?

Geostrophic winds are driven by pressure differences that result from temperature differences. Geostrophic winds occur at altitudes above 1000 m.



Hurricane Bonnie

Extend

19. Normally, expressions such as $x^2 - 5$ are not factored. However, you could express $x^2 - 5$ as $(x - \sqrt{5})(x + \sqrt{5})$.

a) Do you agree that $x^2 - 5$ and $(x - \sqrt{5})(x + \sqrt{5})$ are equivalent? Explain why or why not.

b) Show how factoring could be used to simplify the product $\left(\frac{x + \sqrt{3}}{x^2 - 3}\right)\left(\frac{x^2 - 7}{x - \sqrt{7}}\right)$.

c) What is the simplest form of $\frac{x^2 - 7}{x - \sqrt{7}}$ if you rationalize the denominator? How does this answer compare to the value of $\frac{x^2 - 7}{x - \sqrt{7}}$ that you obtained by factoring in part b)?

20. Fog can be cleared from airports, highways, and harbours using fog-dissipating materials. One device for fog dissipation launches canisters of dry ice to a height defined by $\frac{V^2 \sin x}{2g}$, where V is the exit velocity of the canister, x is the angle of elevation, and g is the acceleration due to gravity.

a) What approximate height is achieved by a canister with $V = 85$ m/s, $x = 52°$, and $g = 9.8$ m/s^2?

b) What height can be achieved if $V = \frac{x + 3}{x - 5}$ metres per second and $x = 30°$?

Fog over Vancouver

Create Connections

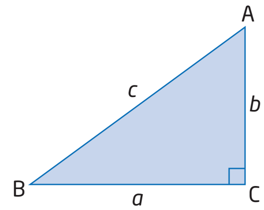
21. Multiplying and dividing rational expressions is very much like multiplying and dividing rational numbers. Do you agree or disagree with this statement? Support your answer with examples.

22. Two points on a coordinate grid are represented by M(p − 1, 2p + 3) and N(2p − 5, p + 1).

a) What is a simplified rational expression for the slope of the line passing through M and N?

b) Write a rational expression for the slope of any line that is perpendicular to MN.

23. Consider △ABC as shown.



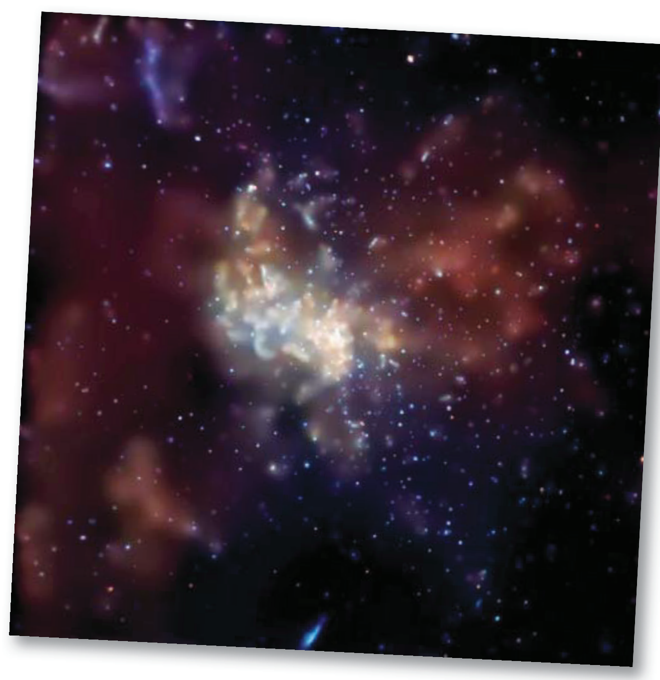
a) What is an expression for tan B?

b) What is an expression for $\frac{\sin B}{\cos B}$? Use your knowledge of rational expressions to help you write the answer in simplest form.

c) How do your expressions for tan B and $\frac{\sin B}{\cos B}$ compare? What can you conclude from this exercise?

Project Corner — Space Anomalies

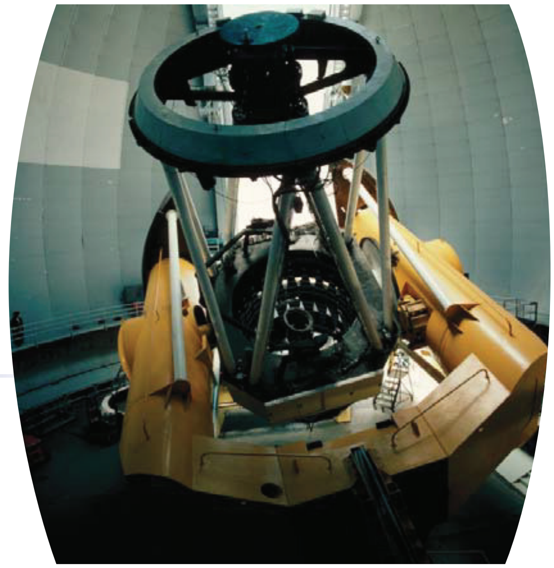
- Time dilation is a phenomenon described by the theory of general relativity. It is the difference in the rate of the passage of time, and it can arise from the relative velocity of motion between observers and the difference in their distance from a gravitational mass.
- A planetary year is the length of time it takes a planet to revolve around the Sun. An Earth year is about 365 days long.
- There is compelling evidence that a super-massive black hole of more than 4 million solar masses is located near the Sagittarius A* region in the centre of the Milky Way galaxy.
- To predict space weather and the effects of solar activity on Earth, an understanding of both solar flares and coronal mass ejections is needed.
- What other types of information about the universe would be useful when predicting the future of space travel?



Black hole near Sagittarius A*

6.3

Adding and Subtracting Rational Expressions



Canada-France-Hawaii Telescope

Focus on…

- connecting addition and subtraction of rational expressions to the same operations with rational numbers
- identifying non-permissible values when adding and subtracting rational expressions
- determining, in simplified form, the sum or difference of rational expressions with the same denominators or with different denominators

Rational expressions are important in photography and in understanding telescopes, microscopes, and cameras. The lens equation can be written as $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$, where f is the focal length, u is the distance from the object to the lens, and v is the distance of the image from the lens. How could you simplify the expression on the right of the lens equation?

Did You Know?

The Canada-France-Hawaii Telescope (CFHT) is a non-profit partnership that operates a 3.6-m telescope atop Mauna Kea in Hawaii. CFHT has played an important role in studying black holes.

Investigate Adding and Subtracting Rational Expressions

1. Determine each sum or difference using diagrams or manipulatives. Describe a pattern that could be used to find the numerical answer. Give answers in lowest terms.

a) $\frac{1}{8} + \frac{5}{8}$ **b)** $\frac{5}{6} - \frac{3}{8}$

2. Use your pattern(s) from step 1 to help you add or subtract each of the following. Express answers in simplest form. Identify any non-permissible values.

a) $\frac{7x + 1}{x} + \frac{5x - 2}{x}$ **b)** $\frac{x}{x - 3} - \frac{3}{x - 3}$ **c)** $\frac{5}{x - 2} - \frac{3}{x + 2}$

3. Substitute numbers for x in step 2a) and c) to see if your answers are reasonable. What value(s) cannot be substituted in each case?

4. Identify similarities and differences between the processes you used in parts b) and c) in step 2.

Reflect and Respond

5. Describe a process you could use to find the answer in simplest form when adding or subtracting rational expressions.

6. Explain how adding and subtracting rational expressions is related to adding and subtracting rational numbers.

Link the Ideas

Adding or Subtracting Rational Expressions

To add or subtract rational expressions, follow procedures similar to those used in adding or subtracting rational numbers.

Case 1: Denominators Are the Same

If two rational expressions have a common denominator, add or subtract the numerators and write the answer as a rational expression with the new numerator over the common denominator.

Case 2: Denominators Are Different

To add or subtract fractions when the denominators are different, you must write equivalent fractions with the same denominator.

$$\begin{aligned}\frac{10}{3x - 12} - \frac{3}{x - 4} &= \frac{10}{3(x - 4)} - \frac{3}{x - 4}\\ &= \frac{10}{3(x - 4)} - \frac{3(3)}{(x - 4)(3)}\\ &= \frac{10 - 9}{3(x - 4)}\\ &= \frac{1}{3(x - 4)}, x \neq 4\end{aligned}$$

Why is it helpful to factor each denominator?

When adding or subtracting rational expressions, you can use any equivalent common denominator. However, it is usually easier to use the lowest common denominator (LCD).

What is the LCD for $\frac{3}{x^2 - 9} + \frac{4}{x^2 - 6x + 9}$?

Why is it easier to use the lowest common denominator? Try it with and without the LCD to compare.

Factor each denominator.

$$\frac{3}{(x - 3)(x + 3)} + \frac{4}{(x - 3)(x - 3)}$$

The LCD must contain the greatest number of any factor that appears in the denominator of either fraction. If a factor appears once in either or both denominators, include it only once. If a factor appears twice in any denominator, include it twice.

The LCD is $(x + 3)(x - 3)(x - 3)$.

Example 1

Add or Subtract Rational Expressions With Common Denominators

Determine each sum or difference. Express each answer in simplest form. Identify all non-permissible values.

a) $\frac{2a}{b} - \frac{a - 1}{b}$

b) $\frac{2x}{x + 4} + \frac{8}{x + 4}$

c) $\frac{x^2}{x - 2} + \frac{3x}{x - 2} - \frac{10}{x - 2}$

Solution

a)
$$\begin{aligned}\frac{2a}{b} - \frac{a - 1}{b} &= \frac{2a - (a - 1)}{b}\\ &= \frac{2a - a + 1}{b}\\ &= \frac{a + 1}{b}, b \neq 0\end{aligned}$$

Why is $a - 1$ placed in brackets?

The non-permissible value is $b = 0$.

b)
$$\begin{aligned}\frac{2x}{x + 4} + \frac{8}{x + 4} &= \frac{2x + 8}{x + 4}\\ &= \frac{2\cancel{(x + 4)}^{1}}{\cancel{x + 4}_{1}}\\ &= 2, x \neq -4\end{aligned}$$

Factor the numerator.

How could you verify this answer?

The non-permissible value is $x = -4$.

c)
$$\begin{aligned}\frac{x^2}{x - 2} + \frac{3x}{x - 2} - \frac{10}{x - 2} &= \frac{x^2 + 3x - 10}{x - 2}\\ &= \frac{\cancel{(x - 2)}^{1}(x + 5)}{\cancel{x - 2}_{1}}\\ &= x + 5, x \neq 2\end{aligned}$$

The non-permissible value is $x = 2$.

Your Turn

Determine each sum or difference. Express each answer in simplest form. Identify all non-permissible values.

a) $\frac{m}{n} - \frac{m + 1}{n}$

b) $\frac{10m - 1}{4m - 3} - \frac{8 - 2m}{4m - 3}$

c) $\frac{2x^2 - x}{(x - 3)(x + 1)} + \frac{3 - 6x}{(x - 3)(x + 1)} - \frac{8}{(x - 3)(x + 1)}$

Example 2

Add or Subtract Rational Expressions With Unlike Denominators

Simplify. Express your answers in simplest form.

a) $\frac{2x}{xy} + \frac{4}{x^2} - 3$, $x \neq 0$, $y \neq 0$

b) $\frac{y^2 - 20}{y^2 - 4} + \frac{y - 2}{y + 2}$, $y \neq \pm 2$

c) $\frac{1 + \frac{1}{x}}{x - \frac{1}{x}}$, $x \neq 0$, $x \neq \pm 1$

Solution

a) The LCD is x^2y. Write each term as an equivalent rational expression with this denominator.

$$\begin{aligned}\frac{2x}{xy} + \frac{4}{x^2} - 3 &= \frac{2x(x)}{xy(x)} + \frac{4(y)}{x^2(y)} - \frac{3(x^2y)}{x^2y}\\ &= \frac{2x^2}{x^2y} + \frac{4y}{x^2y} - \frac{3x^2y}{x^2y}\\ &= \frac{2x^2 + 4y - 3x^2y}{x^2y}\end{aligned}$$

Therefore, $\frac{2x}{xy} + \frac{4}{x^2} - 3 = \frac{2x^2 + 4y - 3x^2y}{x^2y}$, $x \neq 0$, $y \neq 0$

b) Factor the first denominator, and then express each rational expression as an equivalent expression with the common denominator $(y - 2)(y + 2)$.

$$\begin{aligned}&\frac{y^2 - 20}{y^2 - 4} + \frac{y - 2}{y + 2}\\ &= \frac{y^2 - 20}{(y - 2)(y + 2)} + \frac{(y - 2)}{(y + 2)}\\ &= \frac{y^2 - 20}{(y - 2)(y + 2)} + \frac{(y - 2)(y - 2)}{(y + 2)(y - 2)}\\ &= \frac{y^2 - 20 + (y - 2)(y - 2)}{(y - 2)(y + 2)}\\ &= \frac{y^2 - 20 + (y^2 - 4y + 4)}{(y - 2)(y + 2)}\\ &= \frac{y^2 - 20 + y^2 - 4y + 4}{(y - 2)(y + 2)}\\ &= \frac{2y^2 - 4y - 16}{(y - 2)(y + 2)}\\ &= \frac{2(y - 4)\overset{1}{\cancel{(y + 2)}}}{(y - 2)\underset{1}{\cancel{(y + 2)}}}\\ &= \frac{2(y - 4)}{y - 2}\end{aligned}$$

Therefore, $\frac{y^2 - 20}{y^2 - 4} + \frac{y - 2}{y + 2} = \frac{2(y - 4)}{y - 2}$, $y \neq \pm 2$.

c) $\dfrac{1 + \dfrac{1}{x}}{x - \dfrac{1}{x}} = \dfrac{\dfrac{x + 1}{x}}{\dfrac{x^2 - 1}{x}}$

Find a common denominator in both the numerator and the denominator of the complex fraction.

$= \dfrac{x + 1}{x} \div \dfrac{x^2 - 1}{x}$

$= \dfrac{x + 1}{x^2 - 1}$

What was done to arrive at this rational expression?

$= \dfrac{\cancelto{1}{x + 1}}{(x - 1)(\cancelto{1}{x + 1})}$

$= \dfrac{1}{x - 1}$

The complex rational expression could also be simplified by multiplying the numerator and the denominator by *x*, which is the LCD for all the rational expressions. Try this. Which approach do you prefer? Why?

Therefore, $\dfrac{1 + \dfrac{1}{x}}{x - \dfrac{1}{x}} = \dfrac{1}{x - 1}$, $x \neq 0, \pm 1$.

Your Turn

Simplify. What are the non-permissible values?

a) $\dfrac{4}{p^2 - 1} + \dfrac{3}{p + 1}$

b) $\dfrac{x - 1}{x^2 + x - 6} - \dfrac{x - 2}{x^2 + 4x + 3}$

c) $\dfrac{2 - \dfrac{4}{y}}{y - \dfrac{4}{y}}$

Key Ideas

- You can add or subtract rational expressions with the same denominator by adding or subtracting their numerators.

$$\begin{aligned} \frac{2x - 1}{x + 5} - \frac{x - 4}{x + 5} &= \frac{2x - 1 - (x - 4)}{x + 5} \\ &= \frac{2x - 1 - x + 4}{x + 5} \\ &= \frac{x + 3}{x + 5}, \ x \neq -5 \end{aligned}$$

- You can add or subtract rational expressions with unlike denominators after you have written each as an equivalent expression with a common denominator.
- Although more than one common denominator is always possible, it is often easier to use the lowest common denominator (LCD).

Check Your Understanding

Practise

1. Add or subtract. Express answers in simplest form. Identify any non-permissible values.

   a) $\frac{11x}{6} - \frac{4x}{6}$

   b) $\frac{7}{x} + \frac{3}{x}$

   c) $\frac{5t + 3}{10} + \frac{3t + 5}{10}$

   d) $\frac{m^2}{m + 1} + \frac{m}{m + 1}$

   e) $\frac{a^2}{a - 4} - \frac{a}{a - 4} - \frac{12}{a - 4}$

2. Show that x and $\frac{3x - 7}{9} + \frac{6x + 7}{9}$ are equivalent expressions.

3. Simplify. Identify all non-permissible values.

   a) $\frac{1}{(x - 3)(x + 1)} - \frac{4}{(x + 1)}$

   b) $\frac{x - 5}{x^2 + 8x - 20} + \frac{2x + 1}{x^2 - 4}$

4. Identify two common denominators for each question. What is the LCD in each case?

   a) $\frac{x - 3}{6} - \frac{x - 2}{4}$

   b) $\frac{2}{5ay^2} + \frac{3}{10a^2y}$

   c) $\frac{4}{9 - x^2} - \frac{7}{3 + x}$

5. Add or subtract. Give answers in simplest form. Identify all non-permissible values.

   a) $\frac{1}{3a} + \frac{2}{5a}$

   b) $\frac{3}{2x} + \frac{1}{6}$

   c) $4 - \frac{6}{5x}$

   d) $\frac{4z}{xy} - \frac{9x}{yz}$

   e) $\frac{2s}{5t^2} + \frac{1}{10t} - \frac{6}{15t^3}$

   f) $\frac{6xy}{a^2b} - \frac{2x}{ab^2y} + 1$

6. Add or subtract. Give answers in simplest form. Identify all non-permissible values.

   a) $\frac{8}{x^2 - 4} - \frac{5}{x + 2}$

   b) $\frac{1}{x^2 - x - 12} + \frac{3}{x + 3}$

   c) $\frac{3x}{x + 2} - \frac{x}{x - 2}$

   d) $\frac{5}{y + 1} - \frac{1}{y} - \frac{y - 4}{y^2 + y}$

   e) $\frac{2h}{h^2 - 9} + \frac{h}{h^2 + 6h + 9} - \frac{3}{h - 3}$

   f) $\frac{2}{x^2 + x - 6} + \frac{3}{x^3 + 2x^2 - 3x}$

7. Simplify each rational expression, and then add or subtract. Express answers in simplest form. Identify all non-permissible values.

   a) $\frac{3x + 15}{x^2 - 25} + \frac{4x^2 - 1}{2x^2 + 9x - 5}$

   b) $\frac{2x}{x^3 + x^2 - 6x} - \frac{x - 8}{x^2 - 5x - 24}$

   c) $\frac{n + 3}{n^2 - 5n + 6} + \frac{6}{n^2 - 7n + 12}$

   d) $\frac{2w}{w^2 + 5w + 6} - \frac{w - 6}{w^2 + 6w + 8}$

Apply

8. Linda has made an error in simplifying the following. Identify the error and correct the answer.

$$\frac{6}{x - 2} + \frac{4}{x^2 - 4} - \frac{7}{x + 2}$$
$$= \frac{6(x + 2) + 4 - 7(x - 2)}{(x - 2)(x + 2)}$$
$$= \frac{6x + 12 + 4 - 7x - 14}{(x - 2)(x + 2)}$$
$$= \frac{-x + 2}{(x - 2)(x + 2)}$$

9. Can the rational expression $\frac{-x + 5}{(x - 5)(x + 5)}$ be simplified further? Explain.

10. Simplify. State any non-permissible values.

a) $\dfrac{2 - \dfrac{6}{x}}{1 - \dfrac{9}{x^2}}$

b) $\dfrac{\dfrac{3}{2} + \dfrac{3}{t}}{\dfrac{t}{t+6} - \dfrac{1}{t}}$

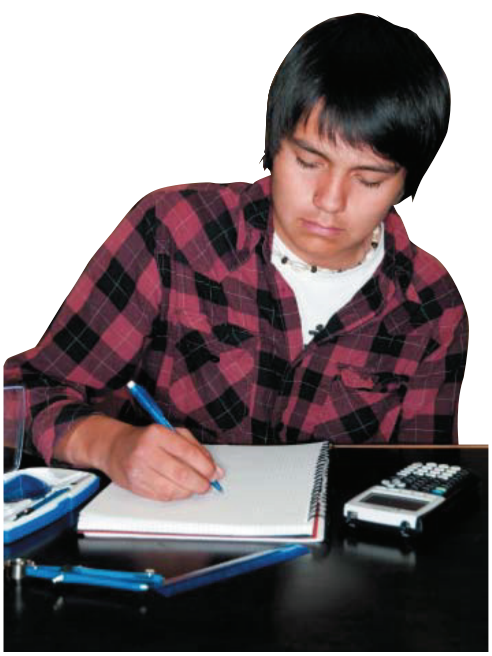
c) $\dfrac{\dfrac{3}{m} - \dfrac{3}{2m+3}}{\dfrac{3}{m^2} + \dfrac{1}{2m+3}}$

d) $\dfrac{\dfrac{1}{x+4} + \dfrac{1}{x-4}}{\dfrac{x}{x^2-16} + \dfrac{1}{x+4}}$

11. Calculators often perform calculations in a different way to accommodate the machine's logic. For each pair of rational expressions, show that the second expression is equivalent to the first one.

a) $\dfrac{A}{B} + \dfrac{C}{D}$; $\dfrac{\dfrac{AD}{B} + C}{D}$

b) $AB + CD + EF$; $\left[\dfrac{\left(\dfrac{AB}{D} + C\right)D}{F} + E\right]F$



12. A right triangle has legs of length $\dfrac{x}{2}$ and $\dfrac{x-1}{4}$. If all measurements are in the same units, what is a simplified expression for the length of the hypotenuse?

13. Ivan is concerned about an underweight calf. He decides to put the calf on a healthy growth program. He expects the calf to gain m kilograms per week and 200 kg in total. However, after some time on the program, Ivan finds that the calf has been gaining $(m + 4)$ kilograms per week.

a) Explain what each of the following rational expressions tells about the situation: $\dfrac{200}{m}$ and $\dfrac{200}{m+4}$.

b) Write an expression that shows the difference between the number of weeks Ivan expected to have the calf on the program and the number of weeks the calf actually took to gain 200 kg.

c) Simplify your rational expression from part b). Does your simplified expression still represent the difference between the expected and actual times the calf took to gain 200 kg? Explain how you know.

14. Suppose you can type an average of n words per minute.

a) What is an expression for the number of minutes it would take to type an assignment with 200 words?

b) Write a sum of rational expressions to represent the time it would take you to type three assignments of 200, 500, and 1000 words, respectively.

c) Simplify the sum in part b). What does the simplified rational expression tell you?

d) Suppose your typing speed decreases by 5 words per minute for each new assignment. Write a rational expression to represent how much longer it would take to type the three assignments. Express your answer in simplest form.

15. Simplify. Identify all non-permissible values.

a) $\frac{x - 2}{x + 5} + \frac{x^2 - 2x - 3}{x^2 - x - 6} \times \frac{x^2 + 2x}{x^2 - 4x}$

b) $\frac{2x^2 - x}{x^2 + 3x} \times \frac{x^2 - x - 12}{2x^2 - 3x + 1} - \frac{x - 1}{x + 2}$

c) $\frac{x - 2}{x + 5} - \frac{x^2 - 2x - 3}{x^2 - x - 6} \times \frac{x^2 + 2x}{x^2 - 4x}$

d) $\frac{x + 1}{x + 6} - \frac{x^2 - 4}{x^2 + 2x} \div \frac{2x^2 + 7x + 3}{2x^2 + x}$

16. A cyclist rode the first 20-kilometre portion of her workout at a constant speed. For the remaining 16-kilometre portion of her workout, she reduced her speed by 2 km/h. Write an algebraic expression for the total time of her bike ride.

17. Create a scenario involving two or more rational expressions. Use your expressions to create a sum or difference. Simplify. Explain what the sum or difference represents in your scenario. Exchange your work with another student in your class. Check whether your classmate's work is correct.

18. Math teachers can identify common errors that students make when adding or subtracting rational expressions. Decide whether each of the following statements is correct or incorrect. Fix each incorrect statement. Indicate how you could avoid making each error.

a) $\frac{a}{b} - \frac{b}{a} = \frac{a - b}{ab}$

b) $\frac{ca + cb}{c + cd} = \frac{a + b}{d}$

c) $\frac{a}{4} - \frac{6 - b}{4} = \frac{a - 6 - b}{4}$

d) $\frac{1}{1 - \frac{a}{b}} = \frac{b}{1 - a}$

e) $\frac{1}{a - b} = \frac{-1}{a + b}$

19. Keander thinks that you can split up a rational expression by reversing the process for adding or subtracting fractions with common denominators. One example is shown below.

$$\begin{aligned}\frac{3x - 7}{x} &= \frac{3x}{x} - \frac{7}{x}\\ &= 3 - \frac{7}{x}\end{aligned}$$

a) Do you agree with Keander? Explain.

b) Keander also claims that by using this method, you can arrive at the rational expressions that were originally added or subtracted. Do you agree or disagree with Keander? Support your decision with several examples.

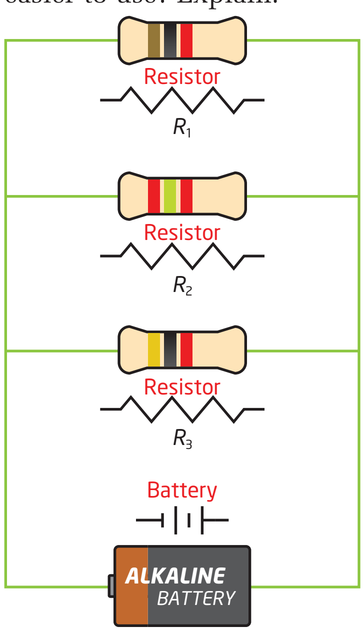
20. A formula for the total resistance, R, in ohms (Ω), of an electric circuit with three resistors in parallel is $R = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}}$, where R_1 is the resistance of the first resistor, R_2 is the resistance of the second resistor, and R_3 is the resistance of the third resistor, all in ohms.

a) What is the total resistance if the resistances of the three resistors are 2 Ω, 3 Ω, and 4 Ω, respectively?

b) Express the right side of the formula in simplest form.

c) Find the total resistance for part a) using your new expression from part b).

d) Which expression for R did you find easier to use? Explain.



Did You Know?

The English physicist James Joule (1818–1889) showed experimentally that the resistance, R, of a resistor can be calculated as $R = \frac{P}{I^2}$, where P is the power, in watts (W), dissipated by the resistor and I is the current, in amperes (A), flowing through the resistor. This is known as Joule's law. A resistor has a resistance of 1 Ω if it dissipates energy at the rate of 1 W when the current is 1 A.

Extend

21. Suppose that $\frac{a}{c} = \frac{b}{d}$, where a, b, c, and d are real numbers. Use both arithmetic and algebra to show that $\frac{a}{c} = \frac{a-b}{c-d}$ is true.

22. Two points on a coordinate grid are represented by $A\left(\frac{p-1}{2}, \frac{p}{3}\right)$ and $B\left(\frac{p}{3}, \frac{2p-3}{4}\right)$.

a) What is a simplified rational expression for the slope of the line passing through A and B?

b) What can you say about the slope of AB when $p = 3$? What does this tell you about the line through A and B?

c) Determine whether the slope of AB is positive or negative when $p < 3$ and p is an integer.

d) Predict whether the slope of AB is positive or negative when $p > 3$ and p is an integer. Check your prediction using $p = 4, 5, 6, \ldots, 10$. What did you find?

23. What is the simplified value of the following expression?

$$\left(\frac{p}{p-x} + \frac{q}{q-x} + \frac{r}{r-x}\right) - \left(\frac{x}{p-x} + \frac{x}{q-x} + \frac{x}{r-x}\right)$$

Create Connections

24. Adding or subtracting rational expressions follows procedures that are similar to those for adding or subtracting rational numbers. Show that this statement is true for expressions with and without common denominators.

25. Two students are asked to find a fraction halfway between two given fractions. After thinking for a short time, one of the students says, "That's easy. Just find the average."

a) Show whether the student's suggestion is correct using arithmetic fractions.

b) Determine a rational expression halfway between $\frac{3}{a}$ and $\frac{7}{2a}$. Simplify your answer. Identify any non-permissible values.

26. Mila claims you can add fractions with the same numerator using a different process.

$$\frac{1}{4} + \frac{1}{3} = \frac{1}{\frac{12}{7}}$$
$$= \frac{7}{12}$$

where the denominator in the first step is calculated as $\frac{4 \times 3}{4 + 3}$.

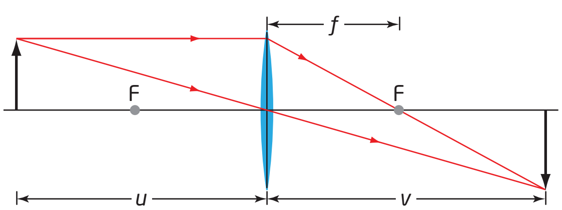
Is Mila's method correct? Explain using arithmetic and algebraic examples.

27. An image found by a convex lens is described by the equation $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$, where f is the focal length (distance from the lens to the focus), u is the distance from the object to the lens, and v is the distance from the image to the lens. All distances are measured in centimetres.

a) Use the mathematical ideas from this section to show that $\frac{1}{f} = \frac{u + v}{uv}$.

b) What is the value of f when $u = 80$ and $v = 6.4$?

c) If you know that $\frac{1}{f} = \frac{u + v}{uv}$, what is a rational expression for f?



28. **MINI LAB** In this section, you have added and subtracted rational expressions to get a single expression. For example, $\frac{3}{x - 4} - \frac{2}{x - 1} = \frac{x + 5}{(x - 4)(x - 1)}$. What if you are given a rational expression and you want to find two expressions that can be added to get it? In other words, you are reversing the situation.

$$\frac{x + 5}{(x - 4)(x - 1)} = \frac{A}{x - 4} + \frac{B}{x - 1}$$

Step 1 To determine A, cover its denominator in the expression on the left, leaving you with $\frac{x + 5}{x - 1}$. Determine the value of $\frac{x + 5}{x - 1}$ when $x = 4$, the non-permissible value for the factor of $x - 4$.

$$\frac{x + 5}{x - 1} = \frac{4 + 5}{4 - 1}$$
$$= 3$$

You can often do this step mentally.

This means $A = 3$.

Step 2 Next, cover $(x - 1)$ in the expression on the left and substitute $x = 1$ into $\frac{x + 5}{x - 4}$ to get -2. This means $B = -2$.

Step 3 Check that it works. Does $\frac{3}{x - 4} - \frac{2}{x - 1} = \frac{x + 5}{(x - 4)(x - 1)}$?

Step 4 What are the values for A and B in the following?

a) $\frac{3x - 1}{(x - 3)(x + 1)} = \frac{A}{x - 3} + \frac{B}{x + 1}$

b) $\frac{6x + 15}{(x + 7)(x - 2)} = \frac{A}{x + 7} + \frac{B}{x - 2}$

Step 5 Do you believe that the method in steps 1 to 3 always works, or sometimes works? Use algebraic reasoning to show that if $\frac{x + 5}{(x - 4)(x - 1)} = \frac{A}{x - 4} + \frac{B}{x - 1}$, then $A = 3$ and $B = -2$.

6.4

Rational Equations


Harbour of ancient Alexandria

Focus on…

- identifying non-permissible values in a rational equation
- determining the solution to a rational equation algebraically
- solving problems using a rational equation

Diophantus of Alexandria is often called the father of new algebra. He is best known for his *Arithmetica*, a work on solving algebraic equations and on the theory of numbers. Diophantus extended numbers to include negatives and was one of the first to describe symbols for exponents. Although it is uncertain when he was born, we can learn his age when he died from the following facts recorded about him:

> … his boyhood lasted $\frac{1}{6}$ of his life; his beard grew after $\frac{1}{12}$ more; he married after $\frac{1}{7}$ more; his son was born 5 years later; the son lived to half his father's age and the father died 4 years later.

How many years did Diophantus live?

Investigate Rational Equations

Work with a partner on the following.

1. An equation can be used to solve the riddle about Diophantus' life. Use x to represent the number of years that he lived.

a) Determine an expression for each unknown part of the riddle. What expression would represent his boyhood, when his beard began to grow, when he married, and so on?

b) How could you represent 5 years later?

c) What is an equation representing his entire life?

2. Begin to solve your equation.

a) What number could you multiply each expression by to make each denominator 1? Perform the multiplication.

b) Solve the resulting linear equation.

c) How old was Diophantus when he died? Check your answer with that of a classmate.

Reflect and Respond

3. Describe a process you could use to solve an equation involving rational expressions.

4. Show similarities and differences between adding and subtracting rational expressions and your process for solving an equation involving them.

Link the Ideas

Solving Rational Equations

rational equation

- an equation containing at least one rational expression
- examples are $x = \frac{x-3}{x+1}$ and $\frac{x}{4} - \frac{7}{x} = 3$

Rational equations can be used to solve several different kinds of problems, such as work-related problems, where two people or machines work together at different rates to complete a task.

Working with a rational equation is similar to working with rational expressions. A significant difference occurs because in an equation, what you do to one side you must also do to the other side.

To solve a rational equation,

- factor each denominator
- identify the non-permissible values
- multiply both sides of the equation by the lowest common denominator
- solve by isolating the variable on one side of the equation
- check your answers

To solve $\frac{x}{4} - \frac{7}{x} = 3$, use the lowest common denominator to express each denominator as 1. The lowest common denominator (LCD) for this equation is $4x$. Proceed by multiplying both sides of the equation by the LCD.

$$4x\left(\frac{x}{4} - \frac{7}{x}\right) = 4x(3),\ x \neq 0$$

$$4x\left(\frac{x}{4}\right) - 4x\left(\frac{7}{x}\right) = 4x(3)$$

Multiply each term on both sides of the equation by $4x$. Simplify each term.

$$x^2 - 28 = 12x$$

$$x^2 - 12x - 28 = 0$$

$$(x - 14)(x + 2) = 0$$

So, $x = 14$ or $x = -2$.

How can you check that these answers are correct?

It is important to realize that non-permissible values are identified from the original equation and that these values cannot be solutions to the final equation.

Example 1

Solve a Rational Equation

Solve the following equation. What values are non-permissible?

$$\frac{2}{z^2 - 4} + \frac{10}{6z + 12} = \frac{1}{z - 2}$$

Solution

$$\frac{2}{z^2 - 4} + \frac{10}{6z + 12} = \frac{1}{z - 2}$$

Factor each denominator.

$$\frac{2}{(z - 2)(z + 2)} + \frac{10}{6(z + 2)} = \frac{1}{z - 2}$$

How can you find the LCD from the factors in the denominators?

From the factors, the non-permissible values are +2 and −2.

$$(z-2)(z+2)(6)\left[\frac{2}{(z-2)(z+2)} + \frac{10}{6(z+2)}\right] = (z-2)(z+2)(6)\left[\frac{1}{z-2}\right]$$

$$\overset{1}{\cancel{(z-2)}}\overset{1}{\cancel{(z+2)}}(6)\left[\frac{2}{\underset{1}{\cancel{(z-2)}}\underset{1}{\cancel{(z+2)}}}\right] + (z-2)\overset{1}{\cancel{(z+2)}}\overset{1}{\cancel{(6)}}\left[\frac{10}{\underset{1}{\cancel{6}}\underset{1}{\cancel{(z+2)}}}\right] = \overset{1}{\cancel{(z-2)}}(z+2)(6)\left(\frac{1}{\underset{1}{\cancel{z-2}}}\right)$$

$$\begin{aligned}(6)(2) + (z-2)(10) &= (z+2)(6)\\ 12 + 10z - 20 &= 6z + 12\\ 4z &= 20\\ z &= 5\end{aligned}$$

What was done in each step?

Check:
Substitute $z = 5$ into the original equation.

Left Side

$$\frac{2}{z^2 - 4} + \frac{10}{6z + 12}$$
$$= \frac{2}{5^2 - 4} + \frac{10}{6(5) + 12}$$
$$= \frac{2}{21} + \frac{10}{42}$$
$$= \frac{2}{21} + \frac{5}{21}$$
$$= \frac{7}{21}$$
$$= \frac{1}{3}$$

Right Side

$$\frac{1}{z - 2}$$
$$= \frac{1}{5 - 2}$$
$$= \frac{1}{3}$$

Left Side = Right Side

The non-permissible values are −2 and 2. The solution cannot be one of the non-permissible values. Since 5 is not one of the non-permissible values, the solution is $z = 5$.

Your Turn

Solve the equation. What are the non-permissible values?

$$\frac{9}{y - 3} - \frac{4}{y - 6} = \frac{18}{y^2 - 9y + 18}$$

Example 2

Solve a Rational Equation With an Extraneous Root

Solve the equation. What are the non-permissible values?

$$\frac{4k-1}{k+2} - \frac{k+1}{k-2} = \frac{k^2-4k+24}{k^2-4}$$

Solution

$$\frac{4k-1}{k+2} - \frac{k+1}{k-2} = \frac{k^2-4k+24}{k^2-4}$$

The factors in the denominators are $k+2$ and $k-2$.

The non-permissible values are 2 and -2.

Describe what is done in the first two steps below.

$$(k-2)(k+2)\left(\frac{4k-1}{k+2} - \frac{k+1}{k-2}\right) = (k-2)(k+2)\left[\frac{k^2-4k+24}{(k-2)(k+2)}\right]$$

$$(k-2)\overset{1}{\cancel{(k+2)}}\left(\frac{4k-1}{\underset{1}{\cancel{k+2}}}\right) - \overset{1}{\cancel{(k-2)}}(k+2)\left(\frac{k+1}{\underset{1}{\cancel{k-2}}}\right) = \overset{1}{\cancel{(k-2)}}\overset{1}{\cancel{(k+2)}}\left[\frac{k^2-4k+24}{\underset{1}{\cancel{(k-2)}}\underset{1}{\cancel{(k+2)}}}\right]$$

$$\begin{aligned}
(k-2)(4k-1)-(k+2)(k+1) &= k^2-4k+24\\
4k^2-9k+2-(k^2+3k+2) &= k^2-4k+24\\
4k^2-9k+2-k^2-3k-2 &= k^2-4k+24\\
2k^2-8k-24 &= 0\\
2(k^2-4k-12) &= 0\\
2(k-6)(k+2) &= 0\\
(k-6)(k+2) &= 0
\end{aligned}$$

So, $k-6=0$ or $k+2=0$.

$k=6$ or $k=-2$

Without further checking, it appears the solutions are -2 and 6. However, -2 is a non-permissible value and is called an extraneous solution.

Check: Substitute $k=6$ into the original equation.

Left Side

$$\begin{aligned}
&\frac{4k-1}{k+2} - \frac{k+1}{k-2}\\
&= \frac{4(6)-1}{6+2} - \frac{6+1}{6-2}\\
&= \frac{23}{8} - \frac{7}{4}\\
&= \frac{23}{8} - \frac{14}{8}\\
&= \frac{9}{8}
\end{aligned}$$

Right Side

$$\begin{aligned}
&\frac{k^2-4k+24}{k^2-4}\\
&= \frac{6^2-4(6)+24}{6^2-4}\\
&= \frac{36}{32}\\
&= \frac{9}{8}
\end{aligned}$$

Left Side = Right Side

Therefore, the solution is $k=6$.

What happens if you check using $k=-2$?

Your Turn

Solve. What are the non-permissible values?

$$\frac{3x}{x+2} - \frac{5}{x-3} = \frac{-25}{x^2-x-6}$$

Example 3

Use a Rational Equation to Solve a Problem

Two friends share a paper route. Sheena can deliver the papers in 40 min. Jeff can cover the same route in 50 min. How long, to the nearest minute, does the paper route take if they work together?

Solution

Make a table to organize the information.

	Time to Deliver Papers (min)	Fraction of Work Done in 1 min	Fraction of Work Done in t minutes
Sheena	40	$\frac{1}{40}$	$\left(\frac{1}{40}\right)(t)$ or $\frac{t}{40}$
Jeff	50	$\frac{1}{50}$	$\frac{t}{50}$
Together	t	$\frac{1}{t}$	$\frac{t}{t}$ or 1

From the table, the equation for Sheena and Jeff to complete the work together is $\frac{t}{40} + \frac{t}{50} = 1$.

Why could this equation also be called a linear equation?

The LCD is 200.

$$200\left(\frac{t}{40}\right) + 200\left(\frac{t}{50}\right) = 200(1)$$
$$5t + 4t = 200$$
$$9t = 200$$
$$t = \frac{200}{9} \text{ or approximately } 22.2$$

Check:
Substitute $t = \frac{200}{9}$ into the original equation.

Left Side

$\frac{t}{40} + \frac{t}{50}$

$= \left(\frac{\frac{200}{9}}{40}\right) + \left(\frac{\frac{200}{9}}{50}\right)$

$= \left(\frac{200}{9}\right)\left(\frac{1}{40}\right) + \left(\frac{200}{9}\right)\left(\frac{1}{50}\right)$

$= \frac{5}{9} + \frac{4}{9}$

$= 1$

Right Side

1

Left Side = Right Side

There are no non-permissible values and the value $t = \frac{200}{9}$ checks. Sheena and Jeff deliver the papers together in approximately 22 min.

Your Turn

Stella takes 4 h to paint a room. It takes Jose 3 h to paint the same area. How long will the paint job take if they work together?

Example 4

Use a Rational Equation to Solve a Problem

The Northern Manitoba Trapper's Festival, held in The Pas, originated in 1916. A championship dog race has always been a significant part of the festivities. In the early days, the race was non-stop from The Pas to Flin Flon and back.

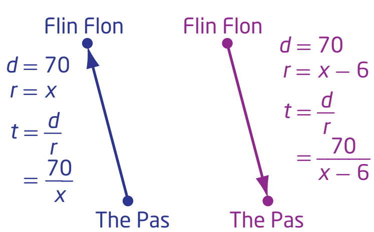
In one particular race, the total distance was 140 mi. Conditions were excellent on the way to Flin Flon. However, bad weather caused the winner's average speed to decrease by 6 mph on the return trip. The total time for the trip was $8\frac{1}{2}$ h. What was the winning dog team's average speed on the way to Flin Flon?

Solution

Use the formula distance = rate × time, or time = $\frac{\text{distance}}{\text{rate}}$.

Let x represent the average speed, in miles per hour, on the trip from The Pas to Flin Flon.

	Distance (mi)	Rate (mph)	Time (h)
Trip to Flin Flon	70	x	$\frac{70}{x}$
Return from Flin Flon	70	$x - 6$	$\frac{70}{x-6}$
		Total	$8\frac{1}{2}$ or $\frac{17}{2}$



$$\frac{70}{x} + \frac{70}{x-6} = \frac{17}{2}$$

What is the LCD for this equation?

What are the non-permissible values?

$$2(x)(x-6)\left(\frac{70}{x} + \frac{70}{x-6}\right) = 2(x)(x-6)\left(\frac{17}{2}\right)$$

$$2\cancel{(x)}^{1}(x-6)\left(\frac{70}{\cancel{x}_{1}}\right) + 2(x)\cancel{(x-6)}^{1}\left(\frac{70}{\cancel{x-6}_{1}}\right) = \cancel{2}^{1}(x)(x-6)\left(\frac{17}{\cancel{2}_{1}}\right)$$

$$2(x-6)(70) + 2(x)(70) = (x)(x-6)(17)$$

$$140x - 840 + 140x = 17x^2 - 102x$$

$$0 = 17x^2 - 382x + 840$$

Use the quadratic formula to solve the equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-382) \pm \sqrt{(-382)^2 - 4(17)(840)}}{2(17)}$$

$$x = \frac{382 \pm \sqrt{88\,804}}{34}$$

$$x = \frac{382 \pm 298}{34}$$

$$x = \frac{382 + 298}{34} \quad \text{or} \quad x = \frac{382 - 298}{34}$$

$$x = 20 \quad \text{or} \quad x = \frac{42}{17}$$

How else might you have solved the quadratic equation? Explain how you know. Try it.

Check: Substitute $x = 20$ and $x = \frac{42}{17}$ into the original equation.

Left Side

$$\frac{70}{x} + \frac{70}{x - 6}$$
$$= \frac{70}{20} + \frac{70}{20 - 6}$$
$$= \frac{7}{2} + \frac{70}{14}$$
$$= 3.5 + 5$$
$$= 8.5$$

Right Side

$$\frac{17}{2} = 8.5$$

Left Side = Right Side

Left Side

$$\frac{70}{x} + \frac{70}{x - 6}$$
$$= \frac{70}{\frac{42}{17}} + \frac{70}{\frac{42}{17} - 6}$$
$$= \frac{70}{\frac{42}{17}} + \frac{70}{\frac{42}{17} - \frac{102}{17}}$$
$$= \frac{70}{\frac{42}{17}} - \frac{70}{\frac{60}{17}}$$
$$= 70\left(\frac{17}{42}\right) - 70\left(\frac{17}{60}\right)$$
$$= \frac{170}{6} - \frac{119}{6}$$
$$= 8.5$$

Right Side

$$\frac{17}{2} = 8.5$$

Left Side = Right Side

Although both solutions have been verified and both are permissible, the solution $\frac{42}{17}$ is inappropriate for this context because if this speed was reduced by 6 mph, then the speed on the return trip would be negative. Therefore, the only solution to the problem is 20 mph.

The winning dog team's average speed going to Flin Flon was 20 mph.

Your Turn

A train has a scheduled run of 160 km between two cities in Saskatchewan. If the average speed is decreased by 16 km/h, the run will take $\frac{1}{2}$ h longer. What is the average speed of the train?

Key Ideas

- You can solve a rational equation by multiplying both sides by a common denominator. This eliminates the fractions from the equation. Then, solve the resulting equation.
- When solving a word problem involving rates, it is helpful to use a table.
- Check that the potential roots satisfy the original equation, are not non-permissible values, and, in the case of a word problem, are realistic in the context.

Check Your Understanding

Practise

1. Use the LCD to eliminate the fractions from each equation. Do not solve.

a) $\dfrac{x-1}{3} - \dfrac{2x-5}{4} = \dfrac{5}{12} + \dfrac{x}{6}$

b) $\dfrac{2x+3}{x+5} + \dfrac{1}{2} = \dfrac{7}{2x+10}$

c) $\dfrac{4x}{x^2-9} - \dfrac{5}{x+3} = 2$

2. Solve and check each equation. Identify all non-permissible values.

a) $\dfrac{f+3}{2} - \dfrac{f-2}{3} = 2$

b) $\dfrac{3-y}{3y} + \dfrac{1}{4} = \dfrac{1}{2y}$

c) $\dfrac{9}{w-3} - \dfrac{4}{w-6} = \dfrac{18}{w^2-9w+18}$

3. Solve each rational equation. Identify all non-permissible values.

a) $\dfrac{6}{t} + \dfrac{t}{2} = 4$

b) $\dfrac{6}{c-3} = \dfrac{c+3}{c^2-9} - 5$

c) $\dfrac{d}{d+4} = \dfrac{2-d}{d^2+3d-4} + \dfrac{1}{d-1}$

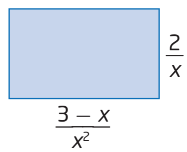
d) $\dfrac{x^2+x+2}{x+1} - x = \dfrac{x^2-5}{x^2-1}$

4. Joline solved the following rational equation. She claims that the solution is $y = 1$. Do you agree? Explain.

$$\frac{-3y}{y-1} + 6 = \frac{6y-9}{y-1}$$

Apply

5. A rectangle has the dimensions shown.



a) What is an expression for the difference between the length and the width of the rectangle? Simplify your answer.

b) What is an expression for the area of the rectangle? Express the answer in simplest form.

c) If the perimeter of the rectangle is 28 cm, find the value(s) for x.

6. Solve. Round answers to the nearest hundredth.

a) $\dfrac{26}{b+5} = 1 + \dfrac{3}{b-2}$

b) $\dfrac{c}{c+2} - 3 = \dfrac{-6}{c^2-4}$

7. Experts claim that the golden rectangle is most pleasing to the eye. It has dimensions that satisfy the equation $\dfrac{l}{w} = \dfrac{l+w}{l}$, where w is the width and l is the length.

According to this relationship, how long should a rectangular picture frame be if its width is 30 cm? Give the exact answer and an approximate answer, rounded to the nearest tenth of a centimetre.

8. The sum of two numbers is 25. The sum of their reciprocals is $\frac{1}{4}$. Determine the two numbers.

9. Two consecutive numbers are represented by x and $x + 1$. If 6 is added to the first number and two is subtracted from the second number, the quotient of the new numbers is $\frac{9}{2}$. Determine the numbers algebraically.

10. A French club collected the same amount from each student going on a trip to Le Cercle Molière in Winnipeg. When six students could not go, each of the remaining students was charged an extra \$3. If the total cost was \$540, how many students went on the trip?

Did You Know?

Le Cercle Molière is the oldest continuously running theatre company in Canada, founded in 1925. It is located in St. Boniface, Manitoba, and moved into its new building in 2009.

11. The sum of the reciprocals of two consecutive integers is $\frac{11}{30}$. What are the integers?

12. Suppose you are running water into a tub. The tub can be filled in 2 min if only the cold tap is used. It fills in 3 min if only the hot tap is turned on. How long will it take to fill the tub if both taps are on simultaneously?

a) Will the answer be less than or greater than 2 min? Why?

b) Complete a table in your notebook similar to the one shown.

	Time to Fill Tub (min)	Fraction Filled in 1 min	Fraction Filled in x minutes
Cold Tap			
Hot Tap			
Both Taps	x	$\frac{1}{x}$	$\frac{x}{x}$ or 1

c) What is one equation that represents both taps filling the tub?

d) Solve your equation to determine the time with both taps running.

13. Two hoses together fill a pool in 2 h. If only hose A is used, the pool fills in 3 h. How long would it take to fill the pool if only hose B were used?

14. Two kayakers paddle 18 km downstream with the current in the same time it takes them to go 8 km upstream against the current. The rate of the current is 3 km/h.

a) Complete a table like the following in your notebook. Use the formula distance = rate × time.

	Distance (km)	Rate (km/h)	Time (h)
Downstream			
Upstream			

b) What equation could you use to find the rate of the kayakers in still water?

c) Solve your equation.

d) Which values are non-permissible?

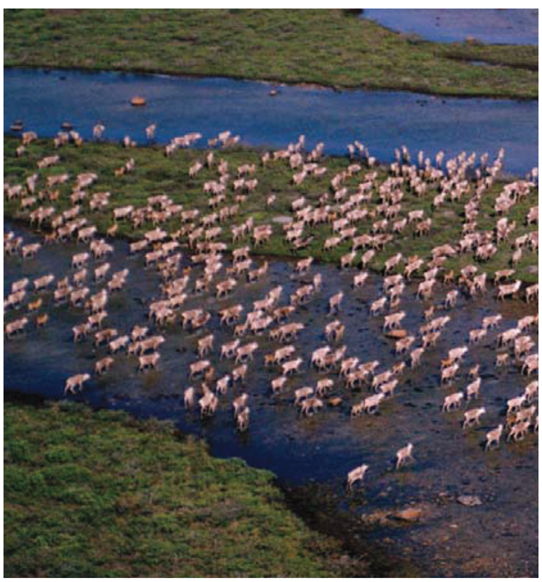
Did You Know?

When you are travelling with the current, add the speed of the current to your rate of speed. When you are travelling against the current, subtract the speed of the current.

Kyuquot Sound, British Columbia

15. Nikita lives in Kindersley, Saskatchewan. With her old combine, she can harvest her entire wheat crop in 72 h. Her neighbour offers to help. His new combine can do the same job in 48 h. How long would it take to harvest the wheat crop with both combines working together?

16. Several cows from the Qamanirjuaq Caribou Herd took 4 days longer to travel 70 km to Forde Lake, Nunavut, than it took them to travel 60 km north beyond Forde Lake. They averaged 5 km/h less before Forde Lake because foraging was better. What was their average speed for the part beyond Forde Lake? Round your answer to the nearest tenth of a kilometre per hour.



Caribou migrating across the tundra, Hudson Bay

Did You Know?

The Qamanirjuaq (ka ma nir you ak) and Beverly barren ground caribou herds winter in the same areas of northern Manitoba, Saskatchewan, Northeast Alberta, and the Northwest Territories and migrate north in the spring into different parts of Nunavut for calving.

Web Link

This is the caribou herd that Inuit depended on in Farley Mowat's book *People of the Deer*. For more information, go to www.mhrprecalc11.ca and follow the links.

17. Ted is a long-distance driver. It took him 30 min longer to drive 275 km on the Trans-Canada Highway west of Swift Current, Saskatchewan, than it took him to drive 300 km east of Swift Current. He averaged 10 km/h less while travelling west of Swift Current due to more severe snow conditions. What was Ted's average speed for each part of the trip?

18. Two friends can paddle a canoe at a rate of 6 km/h in still water. It takes them 1 h to paddle 2 km up a river and back again. Find the speed of the current.

19. Suppose you have 21 days to read a 518-page novel. After finishing half the book, you realize that you must read 12 pages more per day to finish on time. What is your reading rate for the first half of the book? Use a table like the following to help you solve the problem.

	Reading Rate in Pages per Day	Number of Pages Read	Number of Days
First Half			
Second Half			

20. The concentration, C, of salt in a solution is determined by the formula $C = \frac{A}{s + w}$, where A is the constant amount of salt, s is the initial amount of solution, and w is the amount of water added.

a) How much water must be added to a 1-L bottle of 30% salt solution to get a 10% solution?

b) How much water must be added to a half-litre bottle of 10% salt solution to get a 2% solution?

Extend

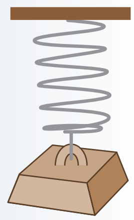
21. If $b = \frac{1}{a}$ and $\dfrac{\frac{1}{a} - \frac{1}{b}}{\frac{1}{a} + \frac{1}{b}} = \frac{4}{5}$, solve for a.

22. A number x is the harmonic mean of a and b if $\frac{1}{x}$ is the average of $\frac{1}{a}$ and $\frac{1}{b}$.

a) Write a rational equation for the statement above. Solve for x.

b) Find two numbers that differ by 8 and have a harmonic mean of 6.

Did You Know?



The distance that a spring stretches is represented by the formula $d = km$, where d is the distance, in centimetres, m is the mass, in grams, and k is a constant.

When two springs with constants k and j are attached to one another, the new spring constant, c, can be found using the formula $\frac{1}{c} = \frac{1}{k} + \frac{1}{j}$. The new spring constant is the harmonic mean of the spring constants for the separate springs.

23. Sometimes is it helpful to solve for a specific variable in a formula. For example, if you solve for x in the equation $\frac{1}{x} - \frac{1}{y} = a$, the answer is $x = \frac{y}{ay + 1}$.

a) Show algebraically how you could solve for x if $\frac{1}{x} - \frac{1}{y} = a$. Show two different ways to get the answer.

b) In the formula $d = v_0t + \frac{1}{2}gt^2$, solve for v_0. Simplify your answer.

c) Solve for n in the formula $I = \dfrac{E}{R + \frac{r}{n}}$.

Create Connections

24. a) Explain the difference between rational expressions and rational equations. Use examples.

b) Explain the process you would use to solve a rational equation. As a model, describe each step you would use to solve the following equation.

$$\frac{5}{x} - \frac{1}{x - 1} = \frac{1}{x - 1}$$

c) There are at least two different ways to begin solving the equation in part b). Identify a different first step from how you began your process in part b).

25. a) A laser printer prints 24 more sheets per minute than an ink-jet printer. If it takes the two printers a total of 14 min to print 490 pages, what is the printing rate of the ink-jet printer?

Did You Know?

Is a paperless office possible? According to Statistics Canada, our consumption of paper for printing and writing more than doubled between 1983 and 2003 to 91 kg, or about 20 000 pages per person per year. This is about 55 pages per person per day.

b) Examine the data in Did You Know? Is your paper use close to the national average? Explain.

c) We can adopt practices to conserve paper use. Work with a partner to identify eco-responsible ways you can conserve paper and ink.

26. Suppose there is a quiz in your mathematics class every week. The value of each quiz is 50 points. After the first 6 weeks, your average mark on these quizzes is 36.

a) What average mark must you receive on the next 4 quizzes so that your average is 40 on the first 10 quizzes? Use a rational equation to solve this problem.

b) There are 15 quizzes in your mathematics course. Show if it is possible to have an average of 90% on your quizzes at the end of the course if your average is 40 out of 50 on the first 10 quizzes.

27. Tyler has begun to solve a rational equation. His work is shown below.

$$\begin{aligned} \frac{2}{x - 1} - 3 &= \frac{5x}{x + 1} \\ 2(x + 1) - 3(x + 1)(x - 1) &= 5x(x - 1) \\ 2x + 2 - 3x^2 + 1 &= 5x^2 - 5x \\ 0 &= 8x^2 - 7x - 3 \end{aligned}$$

a) Re-work the solution to correct any errors that Tyler made.

b) Solve for x. Give your answers as exact values.

c) What are the approximate values of x, to the nearest hundredth?

Chapter 6 Review

6.1 Rational Expressions, pages 310–321

1. A rational number is of the form $\frac{a}{b}$, where a and b are integers.

a) What integer cannot be used for b? Why?

b) How does your answer to part a) relate to rational expressions? Explain using examples.

2. You can write an unlimited number of equivalent expressions for any given rational expression. Do you agree or disagree with this statement? Explain.

3. What are the non-permissible values, if any, for each rational expression?

a) $\frac{5x^2}{2y}$ **b)** $\frac{x^2 - 1}{x + 1}$

c) $\frac{27x^2 - 27}{3}$ **d)** $\frac{7}{(a - 3)(a + 2)}$

e) $\frac{-3m + 1}{2m^2 - m - 3}$ **f)** $\frac{t + 2}{2t^2 - 8}$

4. What is the numerical value for each rational expression? Test your result using some permissible values for the variable. Identify any non-permissible values.

a) $\frac{2s - 8s}{s}$

b) $\frac{5x - 3}{3 - 5x}$

c) $\frac{2 - b}{4b - 8}$

5. Write an expression that satisfies the given conditions in each case.

a) equivalent to $\frac{x - 3}{5}$, with a denominator of $10x$

b) equivalent to $\frac{x - 3}{x^2 - 9}$, with a numerator of 1

c) equivalent to $\frac{c - 2d}{3f}$, with a numerator of $3c - 6d$

d) equivalent to $\frac{m + 1}{m + 4}$, with non-permissible values of ± 4

6. a) Explain how to determine non-permissible values for a rational expression. Use an example in your explanation.

b) Simplify. Determine all non-permissible values for the variables.

i) $\frac{3x^2 - 13x - 10}{3x + 2}$

ii) $\frac{a^2 - 3a}{a^2 - 9}$

iii) $\frac{3y - 3x}{4x - 4y}$

iv) $\frac{81x^2 - 36x + 4}{18x - 4}$

7. A rectangle has area $x^2 - 1$ and width $x - 1$.

a) What is a simplified expression for the length?

b) Identify any non-permissible values. What do they mean in this context?

6.2 Multiplying and Dividing Rational Expressions, pages 322–330

8. Explain how multiplying and dividing rational expressions is similar to multiplying and dividing fractions. Describe how they differ. Use examples to support your response.

9. Simplify each product. Determine all non-permissible values.

a) $\frac{2p}{r} \times \frac{10q}{8p}$

b) $4m^3t \times \frac{1}{16mt^4}$

c) $\frac{3a + 3b}{8} \times \frac{4}{a + b}$

d) $\frac{x^2 - 4}{x^2 + 25} \times \frac{2x^2 + 10x}{x^2 + 2x}$

e) $\frac{d^2 + 3d + 2}{2d + 2} \times \frac{2d + 6}{d^2 + 5d + 6}$

f) $\frac{y^2 - 8y - 9}{y^2 - 10y + 9} \times \frac{y^2 - 9y + 8}{y^2 - 1} \times \frac{y^2 - 25}{5 - y}$

10. Divide. Express answers in simplest form. Identify any non-permissible values.

a) $2t \div \frac{1}{4}$

b) $\frac{a^3}{b^4} \div \frac{a^3}{b^3}$

c) $\frac{7}{x^2 - y^2} \div \frac{-35}{x - y}$

d) $\frac{3a + 9}{a - 3} \div \frac{a^2 + 6a + 9}{a - 3}$

e) $\frac{3x - 2}{x^3 + 3x^2 + 2x} \div \frac{9x^2 - 4}{3x^2 + 8x + 4} \div \frac{1}{x}$

f) $\dfrac{\frac{4 - x^2}{6}}{\frac{x - 2}{2}}$

11. Multiply or divide as indicated. Express answers in simplest form. Determine all non-permissible values.

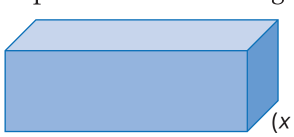
a) $\frac{9}{2m} \div \frac{3}{m} \times \frac{m}{3}$

b) $\frac{x^2 - 3x + 2}{x^2 - 4} \times \frac{x + 3}{x^2 + 3x} \div \frac{1}{x + 2}$

c) $\frac{a - 3}{a - 4} \div \frac{30}{a + 3} \times \frac{5a - 20}{a^2 - 9}$

d) $\frac{3x + 12}{3x^2 - 5x - 12} \times \frac{x - 3}{x + 4} \div \frac{15}{3x + 4}$

12. The volume of a rectangular prism is $(2x^3 + 5x^2 - 12x)$ cubic centimetres. If the length of the prism is $(2x - 3)$ centimetres and its width is $(x + 4)$ centimetres, what is an expression for the height of the prism?



6.3 Adding and Subtracting Rational Expressions, pages 331–340

13. Determine a common denominator for each sum or difference. What is the lowest common denominator (LCD) in each case? What is the advantage of using the LCD?

a) $\frac{4}{5x} + \frac{3}{10x}$

b) $\frac{5}{x - 2} + \frac{2}{x + 1} - \frac{1}{x - 2}$

14. Perform the indicated operations. Express answers in simplest form. Identify any non-permissible values.

a) $\frac{m}{5} + \frac{3}{5}$

b) $\frac{2m}{x} - \frac{m}{x}$

c) $\frac{x}{x + y} + \frac{y}{x + y}$

d) $\frac{x - 2}{3} - \frac{x + 1}{3}$

e) $\frac{x}{x^2 - y^2} - \frac{y}{y^2 - x^2}$

15. Add or subtract. Express answers in simplest form. Identify any non-permissible values.

a) $\frac{4x - 3}{6} - \frac{x - 2}{4}$

b) $\frac{2y - 1}{3y} + \frac{y - 2}{2y} - \frac{y - 8}{6y}$

c) $\frac{9}{x - 3} + \frac{7}{x^2 - 9}$

d) $\frac{a}{a + 3} - \frac{a^2 - 3a}{a^2 + a - 6}$

e) $\frac{a}{a - b} - \frac{2ab}{a^2 - b^2} + \frac{b}{a + b}$

f) $\frac{2x}{4x^2 - 9} + \frac{x}{2x^2 + 5x + 3} - \frac{1}{2x - 3}$

16. The sum of the reciprocals of two numbers will always be the same as the sum of the numbers divided by their product.

a) If the numbers are represented by a and b, translate the sentence above into an equation.

b) Use your knowledge of adding rational expressions to prove that the statement is correct by showing that the left side is equivalent to the right side.

17. Three tests and one exam are given in a course. Let a, b, and c represent the marks of the tests and d be the mark from the final exam. Each is a mark out of 100. In the final mark, the average of the three tests is worth the same amount as the exam. Write a rational expression for the final mark. Show that your expression is equivalent to $\frac{a + b + c + 3d}{6}$. Choose a sample of four marks and show that the simplified expression works.

18. Two sisters go to an auction sale to buy some antique chairs. They intend to pay no more than c dollars for a chair. Beth is worried she will not get the chairs. She bids \$10 more per chair than she intended and spends \$250. Helen is more patient and buys chairs for \$10 less per chair than she intended. She spends \$200 in total.

a) Explain what each of the following expressions represents from the information given about the auction sale.

i) $c + 10$ ii) $c - 10$

iii) $\frac{200}{c - 10}$ iv) $\frac{250}{c + 10}$

v) $\frac{200}{c - 10} + \frac{250}{c + 10}$

b) Determine the sum of the rational expressions in part v) and simplify the result.

6.4 Rational Equations, pages 341–351

19. What is different about the processes used to solve a rational equation from those used to add or subtract rational expressions? Explain using examples.

20. Solve each rational equation. Identify all non-permissible values.

a) $\frac{s - 3}{s + 3} = 2$

b) $\frac{x + 2}{3x + 2} = \frac{x + 3}{x - 1}$

c) $\frac{z - 2}{z} + \frac{1}{5} = \frac{-4}{5z}$

d) $\frac{3m}{m - 3} + 2 = \frac{3m - 1}{m + 3}$

e) $\frac{x}{x - 3} = \frac{3}{x - 3} - 3$

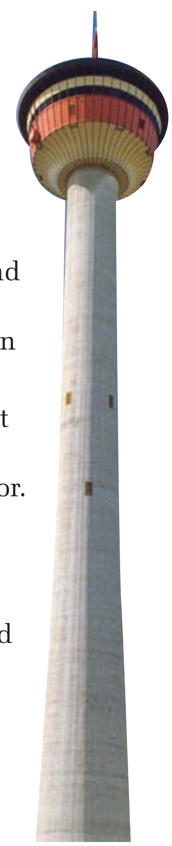
f) $\frac{x - 2}{2x + 1} = \frac{1}{2} + \frac{x - 3}{2x}$

g) $\frac{3}{x + 2} + \frac{5}{x - 3} = \frac{3x}{x^2 - x - 6} - 1$

21. The sum of two numbers is 12. The sum of their reciprocals is $\frac{3}{8}$. What are the numbers?

22. Matt and Elaine, working together, can paint a room in 3 h. It would take Matt 5 h to paint the room by himself. How long would it take Elaine to paint the room by herself?

23. An elevator goes directly from the ground up to the observation deck of the Calgary Tower, which is at 160 m above the ground. The elevator stops at the top for 36 s before it travels directly back down to the ground. The time for the round trip is 2.5 min. The elevator descends at 0.7 m/s faster than it goes up.



a) Determine an equation that could be used to find the rate of ascent of the elevator.

b) Simplify your equation to the form $ax^2 + bx + c = 0$, where a, b, and c are integers, and then solve.

c) What is the rate of ascent in kilometres per hour, to the nearest tenth?

Chapter 6 Practice Test

Multiple Choice

For #1 to #5, choose the best answer.

1. What are the non-permissible values for the rational expression $\frac{x(x + 2)}{(x - 3)(x + 1)}$?

A 0 and -2 **B** -3 and 1
C 0 and 2 **D** 3 and -1

2. Simplify the rational expression $\frac{x^2 - 7x + 6}{x^2 - 2x - 24}$ for all permissible values of x.

A $\frac{x + 1}{x - 4}$ **B** $\frac{x - 1}{x + 4}$
C $\frac{x + 1}{x + 4}$ **D** $\frac{x - 1}{x - 4}$

3. Simplify $\frac{8}{3y} + \frac{5y}{4} - \frac{5}{8}$ for all permissible values of y.

A $\frac{30y^2 - 15y + 64}{24y}$ **B** $\frac{30y^2 + 79}{24y}$
C $\frac{15y^2 + 64}{24y}$ **D** $\frac{5y + 3}{24y}$

4. Simplify $\frac{3x - 12}{9x^2} \div \frac{x - 4}{3x}$, $x \neq 0$ and $x \neq 4$.

A $\frac{1}{x}$ **B** $\frac{16}{3x}$
C x **D** $\frac{-12}{x - 4}$

5. Solve $\frac{6}{t - 3} = \frac{4}{t + 4}$, $t \neq 3$ and $t \neq -4$.

A $-\frac{1}{2}$ **B** -1
C -6 **D** -18

Short Answer

6. Identify all non-permissible values.
$$\frac{3x - 5}{x^2 - 9} \times \frac{2x - 6}{3x^2 - 2x - 5} \div \frac{x - 3}{x + 3}$$

7. If both rational expressions are defined and equivalent, what is the value of k?
$$\frac{2x^2 + kx - 10}{2x^2 + 7x + 6} = \frac{2x - 5}{2x + 3}$$

8. Add or subtract as indicated. Give your answer in simplest form.
$$\frac{5y}{6} + \frac{1}{y - 2} - \frac{y + 1}{3y - 6}$$

9. Create an equation you could use to solve the following problem. Indicate what your variable represents. Do not solve your equation.

A large auger can fill a grain bin in 5 h less time than a smaller auger. Together they fill the bin in 6 h. How long would it take the larger auger, by itself, to fill the bin?

10. List similarities and differences between the processes of adding and subtracting rational expressions and solving rational equations. Use examples.

Extended Response

11. Solve $2 - \frac{5}{x^2 - x - 6} = \frac{x + 3}{x + 2}$. Identify all non-permissible values.

12. The following rational expressions form an arithmetic sequence: $\frac{3 - x}{x}$, $\frac{2x - 1}{2x}$, $\frac{5x + 3}{5x}$. Use common differences to create a rational equation. Solve for x.

13. A plane is flying from Winnipeg to Calgary against a strong headwind of 50 km/h. The plane takes $\frac{1}{2}$ h longer for this flight than it would take in calm air. If the distance from Winnipeg to Calgary is 1200 km, what is the speed of the plane in calm air, to the nearest kilometre per hour?