

Pre-Calculus 11: Chapter 5

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Unit 3

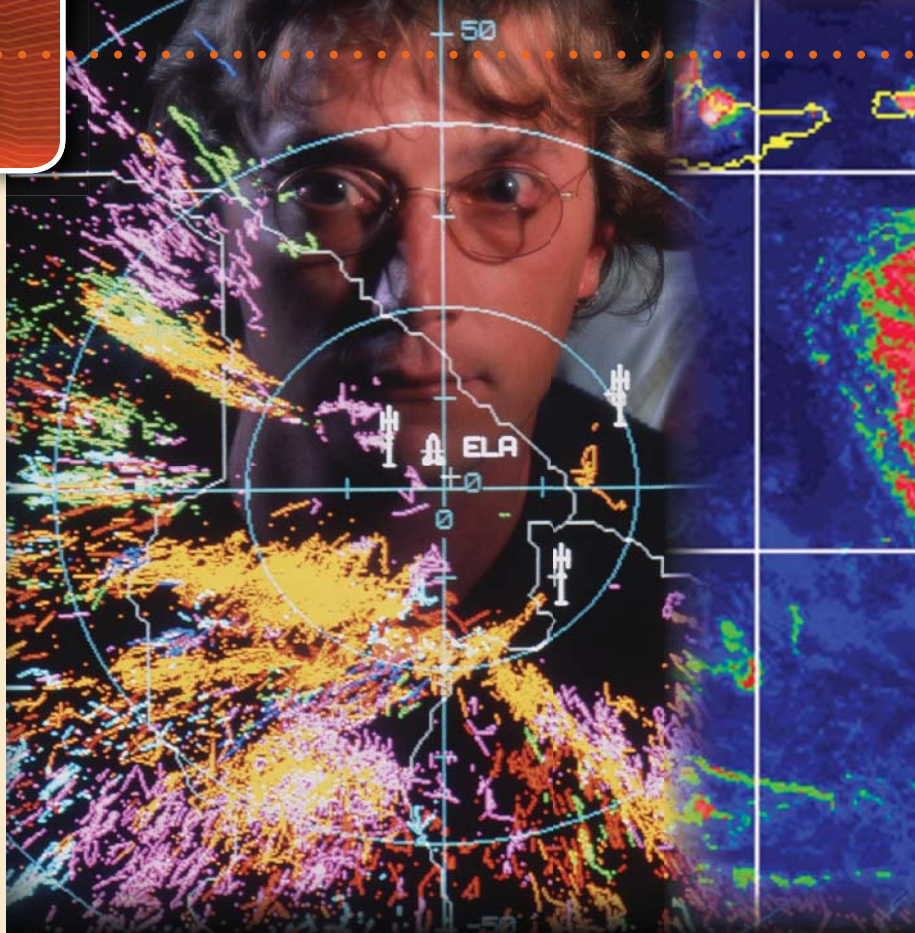
Functions and Equations

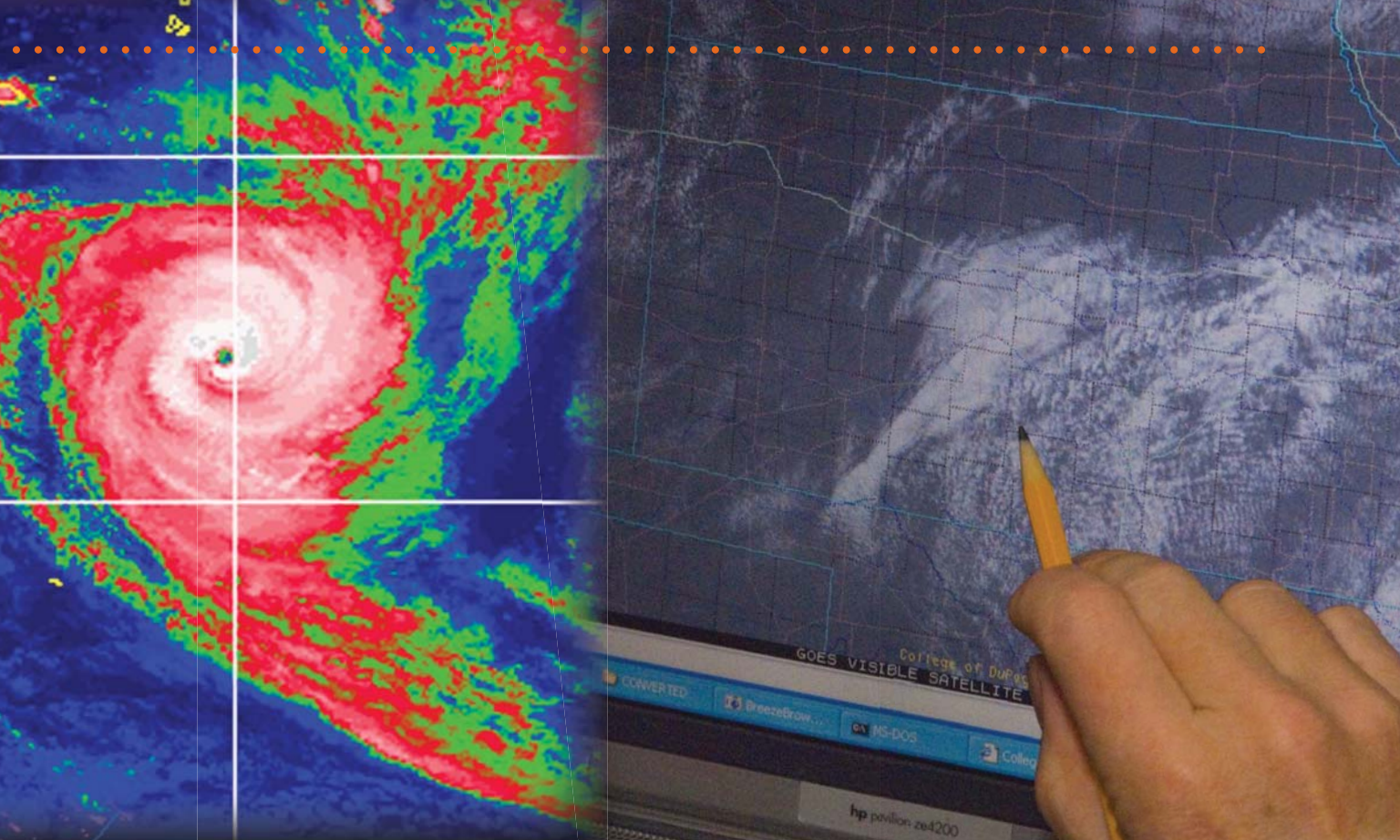
Linear relations have numerous applications in the world. However, mathematicians and scientists have found that many relationships in the natural world cannot be explained with linear models. For example, meteorology, astronomy, and population ecology require more complex mathematical relations to help understand and explain observed phenomena. Similarly, structural engineers and business people need to analyse non-linear data in their everyday working lives. In this unit, you will learn about four types of functions and equations used to model some of the most complex behaviours in our world.

Looking Ahead

In this unit, you will solve problems involving...

- radical expressions and equations
- rational expressions and equations
- absolute value functions and equations
- reciprocal functions





Unit 3 Project

Space: Past, Present, Future

In this project, you will explore a variety of functions and equations, including radical, rational, absolute value, and reciprocal, and how they relate to our understanding of space and its exploration.

In Chapter 5, you will gather information about our galaxy. In Chapter 6, you will gather information about peculiarities in space, such as the passage of time and black holes. In Chapter 7, you will explore space tourism.

At the end of the unit, you will choose at least one of the following three options:

- Examine an application of radicals in space or in the contributions of an astronomer. Investigate why a radical occurs in the mathematics involved in the contribution of the astronomer.
- Research an application of rational expressions in space and investigate why a rational expression models a particular situation.
- Apply the skills you have learned about absolute value functions and reciprocal functions to graphic design.

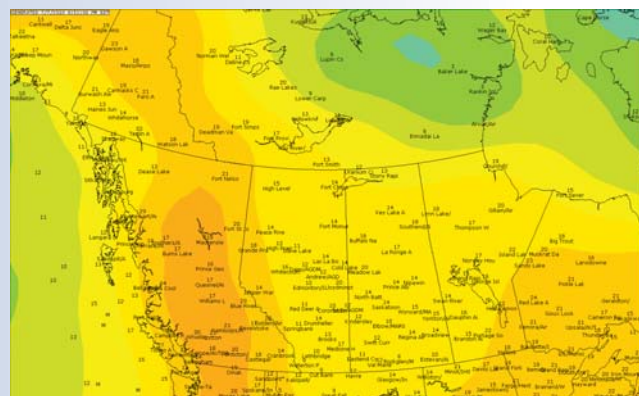
In the Project Corner box at the end of some sections, you will find information and notes about outer space. You can use this information to gather data and facts about your chosen option.

Radical Expressions and Equations

Radical equations can be used to model a variety of relationships—from tracking storms to modelling the path of a football or a skier through the air. Radical expressions and equations allow mathematicians and scientists to work more accurately with numbers. This is important when dealing with large numbers or relations that are sensitive to small adjustments. In this chapter, you will work with a variety of radical expressions and equations including very large radicals as you analyse the cloud formations on the surface of Saturn.

Did You Know?

Weather contour graphs are 3-D graphs that show levels of atmospheric pressure, temperature, precipitation, or ocean heat. The formulas used in these graphs involve squares and square roots. Computers analyse contour graphs of the atmosphere to track weather patterns. Meteorologists use computers and satellite radar to track storms and forecast the weather.

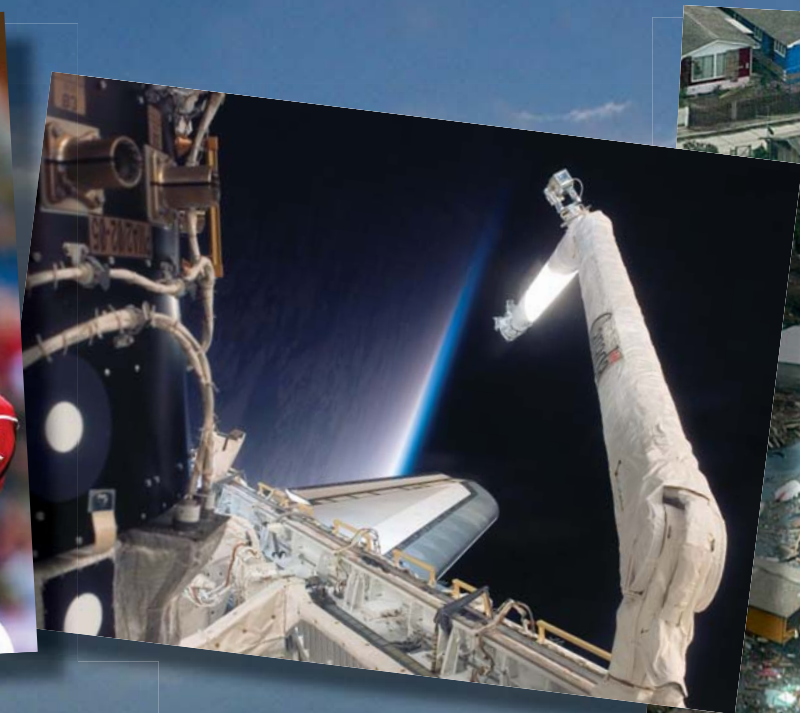


Key Terms

rationalize
conjugates

radical equation





Career Link

Meteorologists study the forces that shape weather and climate. They use formulas that may involve square roots and cube roots to help describe and predict storms and weather patterns. Atmospheric scientists are meteorologists who focus on the atmosphere and investigate the effects of human activities, such as producing pollution, on the atmosphere. Most meteorologists in Canada work for the federal government, and many study at the University of British Columbia or the University of Alberta.

Web Link

To learn more about meteorologists and atmospheric scientists, go to www.mhrprecalc11.ca and follow the links.



Working With Radicals

Focus on...

- converting between mixed radicals and entire radicals
- comparing and ordering radical expressions
- identifying restrictions on the values for a variable in a radical expression
- simplifying radical expressions using addition and subtraction

The packaging industry is huge. It involves design and production, which affect consumers. Graphic designers and packaging engineers apply mathematics skills to designing, constructing, and testing various forms of packaging. From pharmaceuticals to the automobile industry, consumer products are usually found in packages.



Investigate Radical Addition and Subtraction

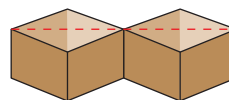
Materials

- 1-cm grid paper
- scissors

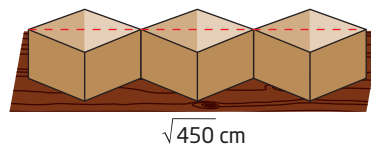
Two glass vases are packaged in opposite corners of a box with a square base. A cardboard divider sits diagonally between the vases. Make a model of the box using grid paper.

- Use an 8 cm by 8 cm square of 1-cm grid paper. Construct a square-based prism without a top. The side length of the base should be double the height of the sides of the box.
 - What is the exact diagonal distance across the base of the model? Explain how you determined the distance.

- The boxes are aligned on display shelves in rows of 2, 4, and 6 boxes each. The boxes are placed corner to corner. What are the exact lengths of the possible rows? Use addition statements to represent your answers. Verify your answers.



- Suppose several classmates place three model boxes along a shelf that is $\sqrt{450}$ cm long.
 - If the boxes are placed side by side, will they fit on the shelf? If so, what distance along the shelf will be occupied? What distance will be unoccupied?
 - Will the boxes fit along the shelf if they are placed corner to corner, with the diagonals forming a straight line? If so, what distance on the shelf will be occupied? What distance will be unoccupied?



4. Write an addition and subtraction statement using only mixed radicals for each calculation in step 3b). A mixed radical is the product of a monomial and a radical. In $r\sqrt[n]{x}$, r is the coefficient, n is the index, and x is the radicand.

Reflect and Respond

5. Develop a general equation that represents the addition of radicals. Compare your equation and method with a classmate's. Identify any rules for using your equation.
6. Use integral values of a to verify that $\sqrt{a} + \sqrt{a} + \sqrt{a} + \sqrt{a} = 4\sqrt{a}$.

Link the Ideas

Like Radicals

Radicals with the same radicand and index are called like radicals.

When adding and subtracting radicals, only like radicals can

be combined. You may need to convert radicals to a different form (mixed or entire) before identifying like radicals.

Pairs of Like Radicals	Pairs of Unlike Radicals
$5\sqrt{7}$ and $-\sqrt{7}$	$2\sqrt{5}$ and $2\sqrt{3}$
$\frac{2}{3}\sqrt[3]{5x^2}$ and $\sqrt[3]{5x^2}$	$\sqrt[4]{5a}$ and $\sqrt[5]{5a}$

How are like radicals similar to like terms?

Restrictions on Variables

If a radical represents a real number and has an even index, the radicand must be non-negative.

The radical $\sqrt{4-x}$ has an even index. So, $4-x$ must be greater than or equal to zero.

$$\begin{aligned}
 4 - x &\geq 0 \\
 4 - x + x &\geq 0 + x \\
 4 &\geq x
 \end{aligned}$$

Isolate the variable by applying algebraic operations to both sides of the inequality symbol.

The radical $\sqrt{4-x}$ is only defined as a real number if x is less than or equal to four. You can check this by substituting values for x that are greater than four, equal to four, and less than four.

Example 1

Convert Mixed Radicals to Entire Radicals

Express each mixed radical in entire radical form. Identify the values of the variable for which the radical represents a real number.

- a) $7\sqrt{2}$ b) $a^4\sqrt{a}$ c) $5b\sqrt[3]{3b^2}$

Solution

- a) Write the coefficient 7 as a square root: $7 = \sqrt{7^2}$.

Then, multiply the radicands of the square roots.

$$\begin{aligned} 7\sqrt{2} &= \sqrt{7^2}(\sqrt{2}) \\ &= \sqrt{7^2(2)} \\ &= \sqrt{49(2)} \\ &= \sqrt{98} \end{aligned}$$

How could you verify the answer?

- b) Express the coefficient a^4 as a square root: $a^4 = \sqrt{(a^4)^2}$.

Multiply the radicals.

$$\begin{aligned} a^4\sqrt{a} &= \sqrt{(a^4)^2}(\sqrt{a}) \\ &= \sqrt{(a^4)^2(a)} \\ &= \sqrt{a^8(a)} \\ &= \sqrt{a^9} \end{aligned}$$

For the radical in the original expression to be a real number, the radicand must be non-negative. Therefore, a is greater than or equal to zero.

- c) Write the entire coefficient, $5b$, as a cube root.

$$\begin{aligned} 5b &= \sqrt[3]{(5b)^3} \\ &= \sqrt[3]{5^3b^3} \end{aligned}$$

Multiply the radicands of the cube roots.

$$\begin{aligned} 5b\sqrt[3]{3b^2} &= (\sqrt[3]{5^3b^3})(\sqrt[3]{3b^2}) \\ &= \sqrt[3]{5^3b^3(3b^2)} \\ &= \sqrt[3]{375b^5} \end{aligned}$$

Why can a radical with an odd index have a radicand that is positive, negative, or zero?

Since the index of the radical is an odd number, the variable, b , can be any real number.

Your Turn

Convert each mixed radical to an entire radical. State the values of the variable for which the radical is a real number.

- a) $4\sqrt{3}$ b) $j^3\sqrt{j}$ c) $2k^2(\sqrt[3]{4k})$

Radicals in Simplest Form

A radical is in simplest form if the following are true.

- The radicand does not contain a fraction or any factor which may be removed.
- The radical is not part of the denominator of a fraction.

For example, $\sqrt{18}$ is not in simplest form because 18 has a square factor of 9, which can be removed. $\sqrt{18}$ is equivalent to the simplified form $3\sqrt{2}$.

Example 2

Express Entire Radicals as Mixed Radicals

Convert each entire radical to a mixed radical in simplest form.

a) $\sqrt{200}$ b) $\sqrt[4]{c^9}$ c) $\sqrt{48y^5}$

Solution

a) Method 1: Use the Greatest Perfect-Square Factor

The following perfect squares are factors of 200: 1, 4, 25, and 100. Write $\sqrt{200}$ as a product using the greatest perfect-square factor.

$$\begin{aligned}\sqrt{200} &= \sqrt{100(2)} \\ &= \sqrt{100}(\sqrt{2}) \\ &= 10\sqrt{2}\end{aligned}$$

Method 2: Use Prime Factorization

Express the radicand as a product of prime factors. The index is two. So, combine pairs of identical factors.

$$\begin{aligned}\sqrt{200} &= \sqrt{2(2)(2)(5)(5)} \\ &= \sqrt{2^2(2)(5^2)} \\ &= 2(5)\sqrt{2} \\ &= 10\sqrt{2}\end{aligned}$$

b) Method 1: Use Prime Factorization

$$\begin{aligned}\sqrt[4]{c^9} &= \sqrt[4]{c(c)(c)(c)(c)(c)(c)(c)} && \text{What number tells you how many} \\ &= \sqrt[4]{c^4(c^4)(c)} && \text{identical factors to combine?} \\ &= c(c)\sqrt[4]{c} \\ &= c^2(\sqrt[4]{c})\end{aligned}$$

Method 2: Use Powers

$$\begin{aligned}\sqrt[4]{c^9} &= c^{\frac{9}{4}} && \text{How will you decide what fractions} \\ &= c^{\frac{8}{4} + \frac{1}{4}} && \text{to use for the sum?} \\ &= c^{\frac{8}{4}}\left(c^{\frac{1}{4}}\right) \\ &= c^2\left(c^{\frac{1}{4}}\right) \\ &= c^2(\sqrt[4]{c})\end{aligned}$$

For the radical to represent a real number, $c \geq 0$ because the index is an even number.

c) $\sqrt{48y^5}$

Determine the greatest perfect-square factors for the numerical and variable parts.

$$\begin{aligned}\sqrt{48y^5} &= \sqrt{16(3)(y^4)(y)} && \text{How can you determine the values of the variables} \\ &= 4y^2\sqrt{3y} && \text{for which the radical is defined as a real number?}\end{aligned}$$

Your Turn

Express each entire radical as a mixed radical in simplest form. Identify any restrictions on the values for the variables.

a) $\sqrt{52}$ b) $\sqrt[4]{m^7}$ c) $\sqrt{63n^7p^4}$

Did You Know?

The radical symbol represents only the *positive* square root. So, even though $(-10)^2 = 100$, $\sqrt{100} \neq \pm 10$. $\sqrt{100} = +10$ $-\sqrt{100} = -10$ In general, $\sqrt{x^2} = x$ only when x is positive.

Did You Know?

A bentwood box is made from a single piece of wood. Heat and moisture make the wood pliable, so it can be bent into a box. First Nations peoples of the Pacific Northwest have traditionally produced these boxes for storage, and some are decorated as symbols of wealth.

Formline Revolution Bentwood Chest
by Corey Moraes, Tsimshian artist



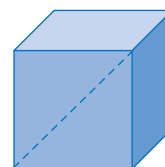
Example 3

Compare and Order Radicals

Five bentwood boxes, each in the shape of a cube have the following diagonal lengths, in centimetres.

$$4(13)^{\frac{1}{2}} \quad 8\sqrt{3} \quad 14 \quad \sqrt{202} \quad 10\sqrt{2}$$

Order the diagonal lengths from least to greatest without using a calculator.



Solution

Express the diagonal lengths as entire radicals.

$$\begin{aligned} 4(13)^{\frac{1}{2}} &= 4\sqrt{13} \\ &= \sqrt{4^2(\sqrt{13})} \\ &= \sqrt{4^2(13)} \\ &= \sqrt{16(13)} \\ &= \sqrt{208} \end{aligned} \qquad \begin{aligned} 8\sqrt{3} &= \sqrt{8^2(\sqrt{3})} \\ &= \sqrt{8^2(3)} \\ &= \sqrt{64(3)} \\ &= \sqrt{192} \end{aligned} \qquad \begin{aligned} 14 &= \sqrt{14^2} \\ &= \sqrt{196} \end{aligned}$$

$$\begin{aligned} \sqrt{202} &\text{ is already} \\ &\text{written as an entire} \\ &\text{radical.} \end{aligned} \qquad \begin{aligned} 10\sqrt{2} &= \sqrt{10^2(\sqrt{2})} \\ &= \sqrt{100(2)} \\ &= \sqrt{200} \end{aligned}$$

Compare the five radicands and order the numbers.

$$\sqrt{192} < \sqrt{196} < \sqrt{200} < \sqrt{202} < \sqrt{208}$$

The diagonal lengths from least to greatest are $8\sqrt{3}$, 14, $10\sqrt{2}$, $\sqrt{202}$, and $4(13)^{\frac{1}{2}}$.

Your Turn

Order the following numbers from least to greatest:

$$5, 3\sqrt{3}, 2\sqrt{6}, \sqrt{23}$$

Example 4

Add and Subtract Radicals

Simplify radicals and combine like terms.

- a) $\sqrt{50} + 3\sqrt{2}$
- b) $-\sqrt{27} + 3\sqrt{5} - \sqrt{80} - 2\sqrt{12}$
- c) $\sqrt{4c} - 4\sqrt{9c}$, $c \geq 0$

Solution

$$\begin{aligned}\text{a) } \sqrt{50} + 3\sqrt{2} &= \sqrt{25(2)} + 3\sqrt{2} \\ &= 5\sqrt{2} + 3\sqrt{2} \\ &= 8\sqrt{2}\end{aligned}$$

How is adding $5\sqrt{2}$ and $3\sqrt{2}$ similar to adding $5x$ and $3x$?

$$\begin{aligned}\text{b) } -\sqrt{27} + 3\sqrt{5} - \sqrt{80} - 2\sqrt{12} \\ &= -\sqrt{3(3)(3)} + 3\sqrt{5} - \sqrt{2(2)(2)(5)} - 2\sqrt{2(2)(3)} \\ &= -3\sqrt{3} + 3\sqrt{5} - 4\sqrt{5} - 4\sqrt{3} \\ &= -7\sqrt{3} - \sqrt{5}\end{aligned}$$

How can you identify which radicals to combine?

$$\begin{aligned}\text{c) } \sqrt{4c} - 4\sqrt{9c} &= \sqrt{4}(\sqrt{c}) - 4\sqrt{9}(\sqrt{c}) \\ &= 2\sqrt{c} - 12\sqrt{c} \\ &= -10\sqrt{c}\end{aligned}$$

Why is $\sqrt{4}$ not equal to ± 2 ?
Why is $\sqrt{9}$ not equal to ± 3 ?

Your Turn

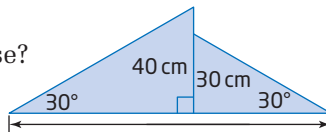
Simplify radicals and combine like terms.

a) $2\sqrt{7} + 13\sqrt{7}$ b) $\sqrt{24} - \sqrt{6}$ c) $\sqrt{20x} - 3\sqrt{45x}$, $x \geq 0$

Example 5

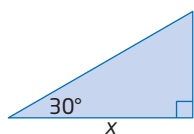
Apply Addition of Radical Expressions

Consider the design shown for a skateboard ramp. What is the exact distance across the base?

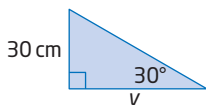


Solution

Redraw each triangle and use trigonometry to determine the lengths, x and y , of the two bases.

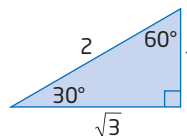


$$\begin{aligned}\tan 30^\circ &= \frac{40}{x} \\ \frac{1}{\sqrt{3}} &= \frac{40}{x} \\ x &= 40\sqrt{3}\end{aligned}$$



$$\begin{aligned}\tan 30^\circ &= \frac{30}{y} \\ \frac{1}{\sqrt{3}} &= \frac{30}{y} \\ y &= 30\sqrt{3}\end{aligned}$$

Recall the ratios of the side lengths of a 30° - 60° - 90° triangle.



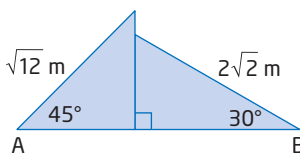
Determine the total length of the bases.

$$\begin{aligned}x + y &= 40\sqrt{3} + 30\sqrt{3} \\ &= 70\sqrt{3}\end{aligned}$$

The distance across the entire base is exactly $70\sqrt{3}$ cm.

Your Turn

What is the exact length of AB?



Key Ideas

- You can compare and order radicals using a variety of strategies:
 - Convert unlike radicals to entire radicals. If the radicals have the same index, the radicands can be compared.
 - Compare the coefficients of like radicals.
 - Compare the indices of radicals with equal radicands.
- When adding or subtracting radicals, combine coefficients of like radicals. In general, $m\sqrt[r]{a} + n\sqrt[r]{a} = (m + n)\sqrt[r]{a}$, where r is a natural number, and m , n , and a are real numbers. If r is even, then $a \geq 0$.
- A radical is in simplest form if the radicand does not contain a fraction or any factor which may be removed, and the radical is not part of the denominator of a fraction.

$$\begin{aligned}\text{For example, } 5\sqrt{40} &= 5\sqrt{4(10)} \\ &= 5\sqrt{4}(\sqrt{10}) \\ &= 5(2)\sqrt{10} \\ &= 10\sqrt{10}\end{aligned}$$

- When a radicand contains variables, identify the values of the variables that make the radical a real number by considering the index and the radicand:
 - If the index is an even number, the radicand must be non-negative.

For example, in $\sqrt{3n}$, the index is even. So, the radicand must be non-negative.

$$3n \geq 0$$

$$n \geq 0$$
 - If the index is an odd number, the radicand may be any real number.

For example, in $\sqrt[3]{x}$, the index is odd. So, the radicand, x , can be any real number—positive, negative, or zero.

Check Your Understanding

Practise

1. Copy and complete the table.

Mixed Radical Form	Entire Radical Form
$4\sqrt{7}$	
	$\sqrt{50}$
$-11\sqrt{8}$	
	$-\sqrt{200}$

2. Express each radical as a mixed radical in simplest form.
- a) $\sqrt{56}$ b) $3\sqrt{75}$
 c) $\sqrt[3]{24}$ d) $\sqrt{c^3d^2}$, $c \geq 0$, $d \geq 0$
3. Write each expression in simplest form. Identify the values of the variable for which the radical represents a real number.
- a) $3\sqrt{8m^4}$ b) $\sqrt[3]{24q^5}$
 c) $-2\sqrt[5]{160s^5t^6}$

4. Copy and complete the table. State the values of the variable for which the radical represents a real number.

Mixed Radical Form	Entire Radical Form
$3n\sqrt{5}$	
	$\sqrt[3]{-432}$
$\frac{1}{2a}\sqrt[3]{7a}$	
	$\sqrt[3]{128x^4}$

5. Express each pair of terms as like radicals. Explain your strategy.
- a) $15\sqrt{5}$ and $8\sqrt{125}$
- b) $8\sqrt{112z^8}$ and $48\sqrt{7z^4}$
- c) $-35\sqrt[4]{w^2}$ and $3\sqrt[4]{81w^{10}}$
- d) $6\sqrt[3]{2}$ and $6\sqrt[3]{54}$
6. Order each set of numbers from least to greatest.

- a) $3\sqrt{6}$, 10, and $7\sqrt{2}$
- b) $-2\sqrt{3}$, -4 , $-3\sqrt{2}$, and $-2\sqrt{\frac{7}{2}}$
- c) $\sqrt[3]{21}$, $3\sqrt[3]{2}$, 2.8, $2\sqrt[3]{5}$

7. Verify your answer to #6b) using a different method.

8. Simplify each expression.

- a) $-\sqrt{5} + 9\sqrt{5} - 4\sqrt{5}$
- b) $1.4\sqrt{2} + 9\sqrt{2} - 7$
- c) $\sqrt[4]{11} - 1 - 5\sqrt[4]{11} + 15$
- d) $-\sqrt{6} + \frac{9}{2}\sqrt{10} - \frac{5}{2}\sqrt{10} + \frac{1}{3}\sqrt{6}$

9. Simplify.

- a) $3\sqrt{75} - \sqrt{27}$
- b) $2\sqrt{18} + 9\sqrt{7} - \sqrt{63}$
- c) $-8\sqrt{45} + 5.1 - \sqrt{80} + 17.4$
- d) $\frac{2}{3}\sqrt[3]{81} + \frac{\sqrt[3]{375}}{4} - 4\sqrt{99} + 5\sqrt{11}$

10. Simplify each expression. Identify any restrictions on the values for the variables.

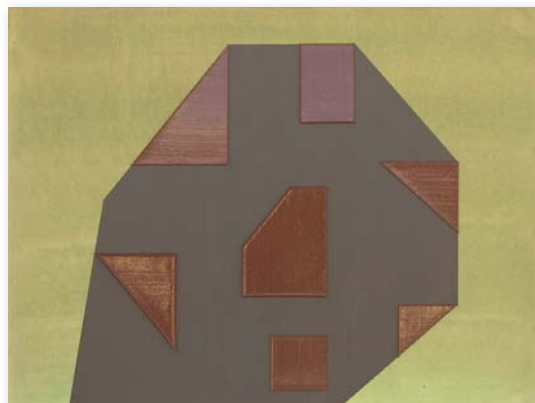
- a) $2\sqrt{a^3} + 6\sqrt{a^3}$
- b) $3\sqrt{2x} + 3\sqrt{8x} - \sqrt{x}$
- c) $-4\sqrt[3]{625r} + \sqrt[3]{40r^4}$
- d) $\frac{w}{5}\sqrt[3]{-64} + \frac{\sqrt[3]{512w^3}}{5} - \frac{2}{5}\sqrt{50w} - 4\sqrt{2w}$

Apply

11. The air pressure, p , in millibars (mbar) at the centre of a hurricane, and wind speed, w , in metres per second, of the hurricane are related by the formula $w = 6.3\sqrt{1013 - p}$. What is the exact wind speed of a hurricane if the air pressure is 965 mbar?



12. Saskatoon artist Jonathan Forrest's painting, *Clincher*, contains geometric shapes. The isosceles right triangle at the top right has legs that measure approximately 12 cm. What is the length of the hypotenuse? Express your answer as a radical in simplest form.



Clincher, by Jonathan Forrest Saskatoon, Saskatchewan

- 13.** The distance, d , in millions of kilometres, between a planet and the Sun is a function of the length, n , in Earth-days, of the planet's year. The formula is $d = \sqrt[3]{25n^2}$. The length of 1 year on Mercury is 88 Earth-days, and the length of 1 year on Mars is 704 Earth-days. Use the subtraction of radicals to determine the difference between the distances of Mercury and Mars from the Sun. Express your answer in exact form.

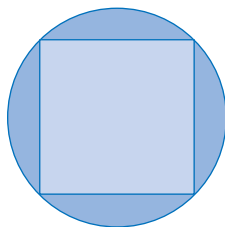


Planet Venus

Did You Know?

Distances in space are frequently measured in astronomical units (AU). The measurement 1 AU represents the average distance between the Sun and Earth during Earth's orbit. According to NASA, the average distance between Venus and Earth is 0.723 AU.

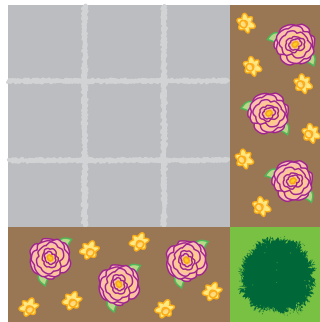
- 14.** The speed, s , in metres per second, of a tsunami is related to the depth, d , in metres, of the water through which it travels. This relationship can be modelled with the formula $s = \sqrt{10d}$, $d \geq 0$. A tsunami has a depth of 12 m. What is the speed as a mixed radical and an approximation to the nearest metre per second?
- 15.** A square is inscribed in a circle. The area of the circle is $38\pi \text{ m}^2$.



$$A_{\text{circle}} = 38\pi \text{ m}^2$$

- What is the exact length of the diagonal of the square?
- Determine the exact perimeter of the square.

- 16.** You can use Heron's formula to determine the area of a triangle given all three side lengths. The formula is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where s represents the half-perimeter of the triangle and a , b , and c are the three side lengths. What is the exact area of a triangle with sides of 8 mm, 10 mm, and 12 mm? Express your answer as an entire radical and as a mixed radical.
- 17.** Suppose an ant travels in a straight line across the Cartesian plane from (3, 4) to (6, 10). Then, it travels in a straight line from (6, 10) to (10, 18). How far does the ant travel? Express your answer in exact form.
- 18.** Leslie's backyard is in the shape of a square. The area of her entire backyard is 98 m^2 . The green square, which contains a tree, has an area of 8 m^2 . What is the exact perimeter of one of the rectangular flowerbeds?



- 19.** Kristen shows her solution to a radical problem below. Brady says that Kristen's final radical is not in simplest form. Is he correct? Explain your reasoning.

Kristen's Solution

$$\begin{aligned} y\sqrt{4y^3} + \sqrt{64y^5} &= y\sqrt{4y^3} + 4y\sqrt{4y^3} \\ &= 5y\sqrt{4y^3} \end{aligned}$$

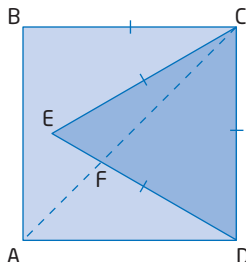
- 20.** Which expression is not equivalent to $12\sqrt{6}$?

$$2\sqrt{216}, 3\sqrt{96}, 4\sqrt{58}, 6\sqrt{24}$$

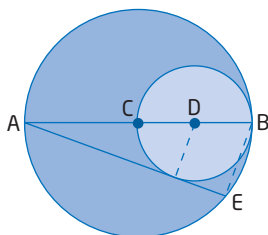
Explain how you know without using technology.

Extend

21. A square, ABCD, has a perimeter of 4 m. $\triangle CDE$ is an equilateral triangle inside the square. The intersection of AC and DE occurs at point F. What is the exact length of AF?



22. A large circle has centre C and diameter AB. A smaller circle has centre D and diameter BC. Chord AE is tangent to the smaller circle. If AB = 18 cm, what is the exact length of AE?



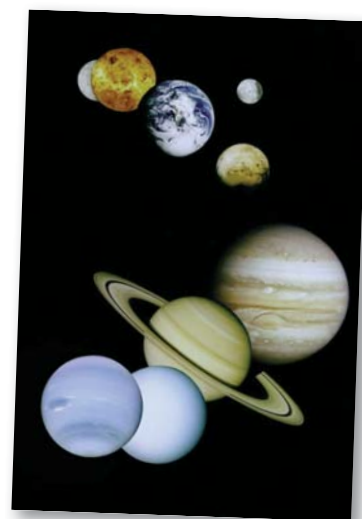
Create Connections

23. What are the exact values of the common difference and missing terms in the following arithmetic sequence? Justify your work.
 $\sqrt{27}$, ■, ■, $9\sqrt{3}$
24. Consider the following set of radicals:
 $-3\sqrt{12}$, $2\sqrt{75}$, $-\sqrt{27}$, $108^{\frac{1}{2}}$
- Explain how you could determine the answer to each question without using a calculator.
- Using only two of the radicals, what is the greatest sum?
 - Using only two of the radicals, what is the greatest difference?
25. Support each equation using examples.
- $(-x)^2 = x^2$
 - $\sqrt{x^2} \neq \pm x$

Project Corner

The Milky Way Galaxy

- Our solar system is located in the Milky Way galaxy, which is a spiral galaxy.
- A galaxy is a congregation of billions of stars, gases, and dust held together by gravity.
- The solar system consists of the Sun, eight planets and their satellites, and thousands of other smaller heavenly bodies such as asteroids, comets, and meteors.
- The motion of planets can be described by Kepler's three laws.
- Kepler's third law states that the ratio of the squares of the orbital periods of any two planets is equal to the ratio of the cubes of their semi-major axes. How could you express Kepler's third law using radicals? Explain this law using words and diagrams.



Multiplying and Dividing Radical Expressions

Focus on...

- performing multiple operations on radical expressions
- rationalizing the denominator
- solving problems that involve radical expressions

In the early 1980s, the Voyager spacecraft first relayed images of a special hexagonal cloud pattern at the north end of Saturn. Due to Saturn's lengthy year (26.4 Earth-years), light has not returned to its north pole until recently. The space probe Cassini has recently returned images of Saturn's north pole. Surprisingly, the hexagonal cloud feature appears to have remained in place over nearly three decades. Scientists are interested in the physics behind this unusual feature of Saturn.



Investigate Radical Multiplication and Division

Materials

- regular hexagon template or compass and ruler to construct regular hexagons
- ruler

Part A: Regular Hexagons and Equilateral Triangles

1. Divide a regular hexagon into six identical equilateral triangles.
2. Suppose the perimeter of the hexagon is 12 cm. Use trigonometry to determine the shortest distance between parallel sides of the hexagon. Express your answer as a mixed radical and an entire radical. What are the angle measures of the triangle you used? Include a labelled diagram.
3. Use another method to verify the distance in step 2.
4. The distance between parallel sides of the hexagonal cloud pattern on Saturn is $\sqrt{468\,750\,000}$ km. Determine the distance, in kilometres, along one edge of the cloud pattern.

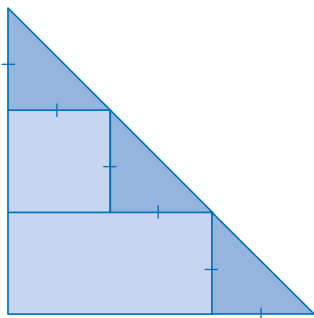
Reflect and Respond

5. Verify your answer to step 4.

Part B: Isosceles Right Triangles and Rectangles

Saturn is more than nine times as far from our Sun as Earth is. At that distance, the Cassini probe is too far from the Sun (1.43 billion kilometres) to use solar panels to operate. However, some spacecrafts and some vehicles do use solar panels to generate power. The vehicle in the photograph uses solar panels and was developed at the University of Calgary.

Consider the following diagram involving rectangular solar panels and isosceles right triangular solar panels.



6. The legs of the three congruent isosceles triangles are $\sqrt{3}$ m long. Determine the dimensions and areas of the two rectangles.
7. What is the exact length of the hypotenuse of the large right triangle? Express your answer in mixed radical form and in entire radical form.
8. Verify your answer to step 7.

Reflect and Respond

9. Consider the two special triangles used in parts A and B. Use trigonometry to relate the angles and ratios of the exact side lengths in lowest terms?
10. Generalize a method for multiplying or dividing any two radicals. Test your method using two examples.
11. Suppose you need to show a classmate how to multiply and divide radicals. What radicals would you use in your example? Why?

Multiplying Radicals

When multiplying radicals, multiply the coefficients and multiply the radicands. You can only multiply radicals if they have the same index.

$$\begin{aligned}
 (2\sqrt{7})(4\sqrt{75}) &= (2)(4)\sqrt{(7)(75)} \\
 &= 8\sqrt{525} \\
 &= 8\sqrt{(25)(21)} && \text{Which method of simplifying a radical is being used?} \\
 &= 8(5)\sqrt{21} \\
 &= 40\sqrt{21}
 \end{aligned}$$

Radicals can be simplified before multiplying:

$$\begin{aligned}
 (2\sqrt{7})(4\sqrt{75}) &= (2\sqrt{7})(4\sqrt{(25)(3)}) \\
 &= (2\sqrt{7})[(4)(5)(\sqrt{3})] \\
 &= (2\sqrt{7})(20\sqrt{3}) \\
 &= (2)(20)\sqrt{(7)(3)} \\
 &= 40\sqrt{21}
 \end{aligned}$$

In general, $(m\sqrt[k]{a})(n\sqrt[k]{b}) = mn\sqrt[k]{ab}$, where k is a natural number, and m , n , a , and b are real numbers. If k is even, then $a \geq 0$ and $b \geq 0$.

Example 1

Multiply Radicals

Multiply. Simplify the products where possible.

- $(-3\sqrt{2x})(4\sqrt{6})$, $x \geq 0$
- $7\sqrt{3}(5\sqrt{5} - 6\sqrt{3})$
- $(8\sqrt{2} - 5)(9\sqrt{5} + 6\sqrt{10})$
- $9\sqrt[3]{2w}(\sqrt[3]{4w} + 7\sqrt[3]{28})$, $w \geq 0$

Solution

$$\begin{aligned}
 \text{a) } (-3\sqrt{2x})(4\sqrt{6}) &= -3(4)\sqrt{(2x)(6)} \\
 &= -12\sqrt{(2x)(2)(3)} \\
 &= -12(2)\sqrt{3x} \\
 &= -24\sqrt{3x}
 \end{aligned}$$

$$\begin{aligned}
 \text{b) } 7\sqrt{3}(5\sqrt{5} - 6\sqrt{3}) &= 7\sqrt{3}(5\sqrt{5}) - 7\sqrt{3}(6\sqrt{3}) && \text{Use the distributive property.} \\
 &= 35\sqrt{15} - 42\sqrt{9} \\
 &= 35\sqrt{15} - 42(3) \\
 &= 35\sqrt{15} - 126
 \end{aligned}$$

$$\begin{aligned}
 \text{c) } (8\sqrt{2} - 5)(9\sqrt{5} + 6\sqrt{10}) &= 8\sqrt{2}(9\sqrt{5}) + 8\sqrt{2}(6\sqrt{10}) - 5(9\sqrt{5}) - 5(6\sqrt{10}) && \text{Use the distributive property.} \\
 &= 72\sqrt{10} + 48\sqrt{20} - 45\sqrt{5} - 30\sqrt{10} && \text{Simplify the radicals.} \\
 &= 72\sqrt{10} + 48\sqrt{(4)(5)} - 45\sqrt{5} - 30\sqrt{10} \\
 &= 72\sqrt{10} + 96\sqrt{5} - 45\sqrt{5} - 30\sqrt{10} && \text{Collect terms with like radicals.} \\
 &= 42\sqrt{10} + 51\sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{d) } 9\sqrt[3]{2w}(\sqrt[3]{4w} + 7\sqrt[3]{28}) &= 9\sqrt[3]{2w(4w)} + 63\sqrt[3]{2w(28)} \\
 &= 9\sqrt[3]{8w^2} + 63\sqrt[3]{56w} \\
 &= 18\sqrt[3]{w^2} + 126\sqrt[3]{7w}
 \end{aligned}$$

Your Turn

Multiply. Simplify where possible.

- a) $5\sqrt{3}(\sqrt{6})$
- b) $-2\sqrt[3]{11}(4\sqrt[3]{2} - 3\sqrt[3]{3})$
- c) $(4\sqrt{2} + 3)(\sqrt{7} - 5\sqrt{14})$
- d) $-2\sqrt{11c}(4\sqrt{2c^3} - 3\sqrt{3}), c \geq 0$

Example 2

Apply Radical Multiplication

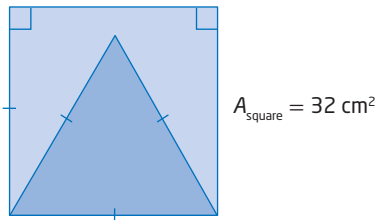
An artist creates a pattern similar to the one shown, but he frames an equilateral triangle inside a square instead of a circle. The area of the square is 32 cm^2 .



- a) What is the exact perimeter of the triangle?
- b) Determine the exact height of the triangle.
- c) What is the exact area of the triangle?
Express all answers in simplest form.

Solution

Create a sketch of the problem.



- a) The side length of the square is $\sqrt{32}$ cm.
Therefore, the base of the triangle is $\sqrt{32}$ cm long.

Simplify the side length.

$$\begin{aligned}
 \sqrt{32} &= \sqrt{16(2)} \\
 &= 4\sqrt{2}
 \end{aligned}$$

How could you determine the greatest perfect-square factor of 32?

Determine the perimeter of the triangle.

$$3(4\sqrt{2}) = 12\sqrt{2}$$

The perimeter of the triangle is $12\sqrt{2}$ cm.

- b) Construct the height, h , in the diagram.

Method 1: Use the Pythagorean Theorem

Since the height bisects the base of the equilateral triangle, there is a right $\triangle ABC$. The lengths of the legs are $2\sqrt{2}$ and h , and the length of the hypotenuse is $4\sqrt{2}$, all in centimetres.

$$\begin{aligned} h^2 + (2\sqrt{2})^2 &= (4\sqrt{2})^2 \\ h^2 + 4(2) &= 16(2) \\ h^2 + 8 &= 32 \\ h^2 &= 24 \\ h &= \pm 2\sqrt{6} \end{aligned}$$

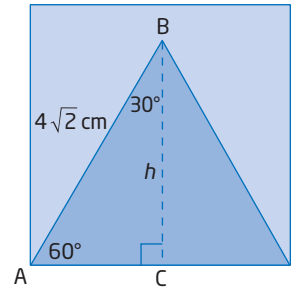
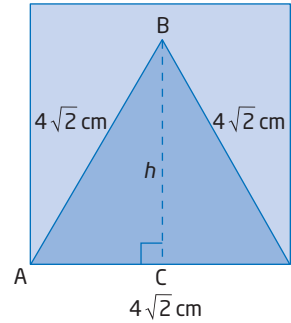
The height of the triangle is $2\sqrt{6}$ cm. Why is only the positive root considered?

Method 2: Use Trigonometry

Identify $\triangle ABC$ as a 30° - 60° - 90° triangle.

$$\begin{aligned} \sin 60^\circ &= \frac{h}{4\sqrt{2}} \\ \frac{\sqrt{3}}{2} &= \frac{h}{4\sqrt{2}} \\ \frac{4\sqrt{2}(\sqrt{3})}{2} &= h \\ 2\sqrt{6} &= h \end{aligned}$$

The height of the triangle is $2\sqrt{6}$ cm.



- c) Use the formula for the area of a triangle, $A = \frac{1}{2}bh$.

$$A = \frac{1}{2}(\sqrt{32})(2\sqrt{6})$$

$$A = \sqrt{(32)(6)}$$

$$A = \sqrt{192}$$

$$A = 8\sqrt{3}$$

The area of the triangle is $8\sqrt{3}$ cm².

Your Turn

An isosceles triangle has a base of $\sqrt{20}$ m. Each of the equal sides is $3\sqrt{7}$ m long. What is the exact area of the triangle?

Dividing Radicals

When dividing radicals, divide the coefficients and then divide the radicands. You can only divide radicals that have the same index.

$$\begin{aligned} \frac{4\sqrt[3]{6}}{2\sqrt[3]{3}} &= 2\sqrt[3]{\frac{6}{3}} \\ &= 2\sqrt[3]{2} \end{aligned}$$

In general, $\frac{m\sqrt[k]{a}}{n\sqrt[k]{b}} = \frac{m}{n}\sqrt[k]{\frac{a}{b}}$, where k is a natural number, and m , n , a , and b are real numbers. $n \neq 0$ and $b \neq 0$. If k is even, then $a \geq 0$ and $b > 0$.

Rationalizing Denominators

To simplify an expression that has a radical in the denominator, you need to **rationalize** the denominator.

For an expression with a monomial square-root denominator, multiply the numerator and denominator by the radical term from the denominator.

$$\begin{aligned}\frac{5}{2\sqrt{3}} &= \frac{5}{2\sqrt{3}} \left(\frac{\sqrt{3}}{\sqrt{3}} \right) && \text{Why is this product equivalent to the original expression?} \\ &= \frac{5\sqrt{3}}{2\sqrt{3}(\sqrt{3})} \\ &= \frac{5\sqrt{3}}{6}\end{aligned}$$

For a binomial denominator that contains a square root, multiply both the numerator and denominator by a **conjugate** of the denominator.

The product of a pair of conjugates is a difference of squares.
 $(a - b)(a + b) = a^2 - b^2$
 $(\sqrt{u} + \sqrt{v})(\sqrt{u} - \sqrt{v}) = (\sqrt{u})^2 + (\sqrt{v})(\sqrt{u}) - (\sqrt{v})(\sqrt{u}) - (\sqrt{v})^2$
 $= u - v$

In the radical expression, $\frac{5\sqrt{3}}{4 - \sqrt{6}}$, the conjugates of $4 - \sqrt{6}$ are $4 + \sqrt{6}$ and $-4 - \sqrt{6}$. If you multiply either of these expressions with the denominator, the product will be a rational number.

$$\begin{aligned}\frac{5\sqrt{3}}{4 - \sqrt{6}} &= \left(\frac{5\sqrt{3}}{4 - \sqrt{6}} \right) \left(\frac{4 + \sqrt{6}}{4 + \sqrt{6}} \right) \\ &= \frac{20\sqrt{3} + 5\sqrt{18}}{4^2 - (\sqrt{6})^2} \\ &= \frac{20\sqrt{3} + 5\sqrt{9(2)}}{16 - 6} \\ &= \frac{20\sqrt{3} + 15\sqrt{2}}{10} \\ &= \frac{4\sqrt{3} + 3\sqrt{2}}{2} && \text{Express in simplest form.}\end{aligned}$$

rationalize

- convert to a rational number without changing the value of the expression
- If the radical is in the denominator, both the numerator and denominator must be multiplied by a quantity that will produce a rational denominator.

conjugates

- two binomial factors whose product is the difference of two squares
- the binomials $(a + b)$ and $(a - b)$ are conjugates since their product is $a^2 - b^2$

Example 3

Divide Radicals

Simplify each expression.

a) $\frac{\sqrt{24x^2}}{\sqrt{3x}}, x > 0$

b) $\frac{4\sqrt{5n}}{3\sqrt{2}}, n \geq 0$

c) $\frac{11}{\sqrt{5} + 7}$

d) $\frac{4\sqrt{11}}{y^3\sqrt{6}}, y \neq 0$

Solution

$$\begin{aligned}\text{a) } \frac{\sqrt{24x^2}}{\sqrt{3x}} &= \sqrt{\frac{24x^2}{3x}} \\ &= \sqrt{8x} \\ &= 2\sqrt{2x}\end{aligned}$$

$$\begin{aligned}\text{b) } \frac{4\sqrt{5n}}{3\sqrt{2}} &= \frac{4\sqrt{5n}}{3\sqrt{2}} \left(\frac{\sqrt{2}}{\sqrt{2}} \right) \\ &= \frac{4\sqrt{10n}}{3(2)} \\ &= \frac{2\sqrt{10n}}{3}\end{aligned}$$

Rationalize the denominator.

$$\begin{aligned}\text{c) } \frac{11}{\sqrt{5} + 7} &= \left(\frac{11}{\sqrt{5} + 7} \right) \left(\frac{\sqrt{5} - 7}{\sqrt{5} - 7} \right) \\ &= \frac{11(\sqrt{5} - 7)}{(\sqrt{5})^2 - 7^2} \\ &= \frac{11(\sqrt{5} - 7)}{5 - 49} \\ &= \frac{11(\sqrt{5} - 7)}{-44} \\ &= \frac{-(\sqrt{5} - 7)}{4} \\ &= \frac{7 - \sqrt{5}}{4}\end{aligned}$$

How do you determine a conjugate of $\sqrt{5} + 7$?

The solution can be verified using decimal approximations.

Initial expression:

$$\frac{11}{\sqrt{5} + 7} \approx 1.19098$$

Final expression:

$$\frac{7 - \sqrt{5}}{4} \approx 1.19098$$

$$\begin{aligned}\text{d) } \frac{4\sqrt{11}}{y\sqrt[3]{6}} &= \frac{4\sqrt{11}}{y\sqrt[3]{6}} \left(\frac{(\sqrt[3]{6})^2}{(\sqrt[3]{6})^2} \right) \\ &= \frac{4\sqrt{11}(\sqrt[3]{6})(\sqrt[3]{6})}{y\sqrt[3]{6}(\sqrt[3]{6})(\sqrt[3]{6})} \\ &= \frac{4\sqrt{11}(\sqrt[3]{36})}{y(6)} \\ &= \frac{2\sqrt{11}(\sqrt[3]{36})}{3y}\end{aligned}$$

How does the index help you determine what expression to use when rationalizing the denominator?

Your Turn

Simplify each quotient. Identify the values of the variable for which the expression is a real number.

$$\text{a) } \frac{2\sqrt{51}}{\sqrt{3}}$$

$$\text{b) } \frac{-7}{2\sqrt[3]{9p}}$$

$$\text{c) } \frac{2}{3\sqrt{5} - 4}$$

$$\text{d) } \frac{6}{\sqrt{4x} + 1}$$

Key Ideas

- When multiplying radicals with identical indices, multiply the coefficients and multiply the radicands:

$$(m\sqrt[k]{a})(n\sqrt[k]{b}) = mn\sqrt[k]{ab}$$

where k is a natural number, and m , n , a , and b are real numbers.

If k is even, then $a \geq 0$ and $b \geq 0$.

- When dividing two radicals with identical indices, divide the coefficients and divide the radicands:

$$\frac{m\sqrt[k]{a}}{n\sqrt[k]{b}} = \frac{m}{n}\sqrt[k]{\frac{a}{b}}$$

where k is a natural number, and m , n , a , and b are real numbers.

$n \neq 0$ and $b \neq 0$. If k is even, then $a \geq 0$ and $b > 0$.

- When multiplying radical expressions with more than one term, use the distributive property and then simplify.
- To rationalize a monomial denominator, multiply the numerator and denominator by an expression that produces a rational number in the denominator.

$$\frac{2}{\sqrt[5]{n}} \left(\frac{(\sqrt[5]{n})^4}{(\sqrt[5]{n})^4} \right) = \frac{2(\sqrt[5]{n})^4}{n}$$

- To simplify an expression with a square-root binomial in the denominator, rationalize the denominator using these steps:
 - Determine a conjugate of the denominator.
 - Multiply the numerator and denominator by this conjugate.
 - Express in simplest form.

Check Your Understanding

Practise

- Multiply. Express all products in simplest form.
 - $2\sqrt{5}(7\sqrt{3})$
 - $-\sqrt{32}(7\sqrt{2})$
 - $2\sqrt[4]{48}(\sqrt[4]{5})$
 - $4\sqrt{19x}(\sqrt{2x^2})$, $x \geq 0$
 - $\sqrt[3]{54y^7}(\sqrt[3]{6y^4})$
 - $\sqrt{6t}\left(3t^2\sqrt{\frac{t}{4}}\right)$, $t \geq 0$
- Multiply using the distributive property. Then, simplify.
 - $\sqrt{11}(3 - 4\sqrt{7})$
 - $-\sqrt{2}(14\sqrt{5} + 3\sqrt{6} - \sqrt{13})$
 - $\sqrt{y}(2\sqrt{y} + 1)$, $y \geq 0$
 - $z\sqrt{3}(z\sqrt{12} - 5z + 2)$
- Simplify. Identify the values of the variables for which the radicals represent real numbers.
 - $-3(\sqrt{2} - 4) + 9\sqrt{2}$
 - $7(-1 - 2\sqrt{6}) + 5\sqrt{6} + 8$
 - $4\sqrt{5}(\sqrt{3j} + 8) - 3\sqrt{15j} + \sqrt{5}$
 - $3 - \sqrt[3]{4k}(12 + 2\sqrt[3]{8})$

4. Expand and simplify each expression.

- a) $(8\sqrt{7} + 2)(\sqrt{2} - 3)$
- b) $(4 - 9\sqrt{5})(4 + 9\sqrt{5})$
- c) $(\sqrt{3} + 2\sqrt{15})(\sqrt{3} - \sqrt{15})$
- d) $(6\sqrt[3]{2} - 4\sqrt{13})^2$
- e) $(-\sqrt{6} + 2)(2\sqrt{2} - 3\sqrt{5} + 1)$

5. Expand and simplify. State any restrictions on the values for the variables.

- a) $(15\sqrt{c} + 2)(\sqrt{2c} - 6)$
- b) $(1 - 10\sqrt{8x^3})(2 + 7\sqrt{5x})$
- c) $(9\sqrt{2m} - 4\sqrt{6m})^2$
- d) $(10r - 4\sqrt[3]{4r})(2\sqrt[3]{6r^2} + 3\sqrt[3]{12r})$

6. Divide. Express your answers in simplest form.

- a) $\frac{\sqrt{80}}{\sqrt{10}}$
- b) $\frac{-2\sqrt{12}}{4\sqrt{3}}$
- c) $\frac{3\sqrt{22}}{\sqrt{11}}$
- d) $\frac{3\sqrt{135m^5}}{\sqrt{21m^3}}, m > 0$

7. Simplify.

- a) $\frac{9\sqrt{432p^5} - 7\sqrt{27p^5}}{\sqrt{33p^4}}, p > 0$
- b) $\frac{6\sqrt[3]{4v^7}}{\sqrt[3]{14v}}, v > 0$

8. Rationalize each denominator. Express each radical in simplest form.

- a) $\frac{20}{\sqrt{10}}$
- b) $\frac{-\sqrt{21}}{\sqrt{7m}}, m > 0$
- c) $-\frac{2}{3}\sqrt{\frac{5}{12u}}, u > 0$
- d) $20\sqrt[3]{\frac{6t}{5}}$

9. Determine a conjugate for each binomial. What is the product of each pair of conjugates?

- a) $2\sqrt{3} + 1$
- b) $7 - \sqrt{11}$
- c) $8\sqrt{z} - 3\sqrt{7}, z \geq 0$
- d) $19\sqrt{h} + 4\sqrt{2h}, h \geq 0$

10. Rationalize each denominator. Simplify.

- a) $\frac{5}{2 - \sqrt{3}}$
- b) $\frac{7\sqrt{2}}{\sqrt{6} + 8}$
- c) $\frac{-\sqrt{7}}{\sqrt{5} - 2\sqrt{2}}$
- d) $\frac{\sqrt{3} + \sqrt{13}}{\sqrt{3} - \sqrt{13}}$

11. Write each fraction in simplest form. Identify the values of the variables for which each fraction is a real number.

- a) $\frac{4r}{\sqrt{6r} + 9}$
- b) $\frac{18\sqrt{3n}}{\sqrt{24n}}$
- c) $\frac{8}{4 - \sqrt{6t}}$
- d) $\frac{5\sqrt{3y}}{\sqrt{10} + 2}$

12. Use the distributive property to simplify $(c + c\sqrt{c})(c + 7\sqrt{3c}), c \geq 0$.

Apply

13. Malcolm tries to rationalize the denominator in the expression

$$\frac{4}{3 - 2\sqrt{2}} \text{ as shown below.}$$

- a) Identify, explain, and correct any errors.
- b) Verify your corrected solution.

Malcolm's solution:

$$\begin{aligned} \frac{4}{3 - 2\sqrt{2}} &= \left(\frac{4}{3 - 2\sqrt{2}} \right) \left(\frac{3 + 2\sqrt{2}}{3 + 2\sqrt{2}} \right) \\ &= \frac{12 + 8\sqrt{4(2)}}{9 - 8} \\ &= 12 + 16\sqrt{2} \end{aligned}$$

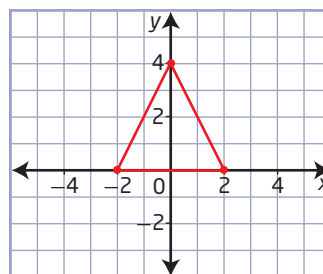
14. In a golden rectangle, the ratio of the side dimensions is $\frac{2}{\sqrt{5}-1}$. Determine an equivalent expression with a rational denominator.
15. The period, T , in seconds, of a pendulum is related to its length, L , in metres. The period is the time to complete one full cycle and can be approximated with the formula $T = 2\pi\sqrt{\frac{L}{10}}$.
- Write an equivalent formula with a rational denominator.
 - The length of the pendulum in the HSBC building in downtown Vancouver is 27 m. How long would the pendulum take to complete 3 cycles?



Did You Know?

The pendulum in the HSBC building in downtown Vancouver has a mass of approximately 1600 kg. It is made from buffed aluminum and is assisted at the top by a hydraulic mechanical system.

16. Jonasie and Iblauk are planning a skidoo race for their community of Uqsuqtuuq or Gjoa Haven, Nunavut. They sketch the triangular course on a Cartesian plane. The area of 1 grid square represents 9245 m². What is the exact length of the red track?

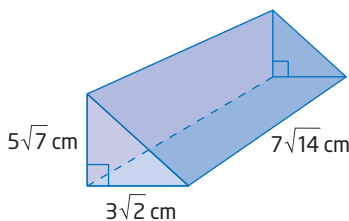


17. Simplify $\left(\frac{1+\sqrt{5}}{2-\sqrt{3}}\right)\left(\frac{1-\sqrt{5}}{2-\sqrt{3}}\right)$.
18. In a scale model of a cube, the ratio of the volume of the model to the volume of the cube is 1:4. Express your answers to each of the following questions as mixed radicals in simplest form.
- What is the edge length of the actual cube if its volume is 192 mm³?
 - What is the edge length of the model cube?
 - What is the ratio of the edge length of the actual cube to the edge length of the model cube?
19. Lev simplifies the expression, $\frac{2x\sqrt{14}}{\sqrt{3}-5x}$. He determines the restrictions on the values for x as follows:
- $$\begin{aligned} 3-5x &> 0 \\ -5x &> -3 \\ x &> \frac{3}{5} \end{aligned}$$
- Identify, explain, and correct any errors.
 - Why do variables involved in radical expressions sometimes have restrictions on their values?
 - Create an expression involving radicals that does not have any restrictions. Justify your response.

20. Olivia simplifies the following expression. Identify, explain, and correct any errors in her work.

$$\begin{aligned}\frac{2c - c\sqrt{25}}{\sqrt{3}} &= \frac{(2c - c\sqrt{25})\left(\frac{\sqrt{3}}{\sqrt{3}}\right)}{\sqrt{3}} \\ &= \frac{\sqrt{3}(2c - c\sqrt{25})}{3} \\ &= \frac{\sqrt{3}(2c \pm 5c)}{3} \\ &= \frac{\sqrt{3}(7c)}{3} \quad \text{or} \quad \frac{\sqrt{3}(-3c)}{3} \\ &= \frac{7c\sqrt{3}}{3} \quad \text{or} \quad -c\sqrt{3}\end{aligned}$$

21. What is the volume of the right triangular prism?



Extend

22. A cube is inscribed in a sphere with radius 1 m. What is the surface area of the cube?
23. Line segment AB has endpoints $A(\sqrt{27}, -\sqrt{50})$ and $B(3\sqrt{48}, 2\sqrt{98})$. What is the midpoint of AB?
24. Rationalize the denominator of $(3(\sqrt{x})^{-1} - 5)^{-2}$. Simplify the expression.
25. a) What are the exact roots of the quadratic equation $x^2 + 6x + 3 = 0$?
 b) What is the sum of the two roots from part a)?
 c) What is the product of the two roots?
 d) How are your answers from parts b) and c) related to the original equation?
26. Rationalize the denominator of $\frac{\sqrt[n]{a}}{\sqrt[n]{t}}$.
27. What is the exact surface area of the right triangular prism in #21?

Create Connections

28. Describe the similarities and differences between multiplying and dividing radical expressions and multiplying and dividing polynomial expressions.
29. How is rationalizing a square-root binomial denominator related to the factors of a difference of squares? Explain, using an example.
30. A snowboarder departs from a jump. The quadratic function that approximately relates height above landing area, h , in metres, and time in air, t , in seconds, is $h(t) = -5t^2 + 10t + 3$.
- What is the snowboarder's height above the landing area at the beginning of the jump?
 - Complete the square of the expression on the right to express the function in vertex form. Isolate the variable t .
 - Determine the exact height of the snowboarder halfway through the jump.



Did You Know?

Maele Ricker from North Vancouver won a gold medal at the 2010 Vancouver Olympics in snowboard cross. She is the first Canadian woman to win a gold medal at a Canadian Olympics.

31. Are $m = \frac{-5 + \sqrt{13}}{6}$ and $m = \frac{-5 - \sqrt{13}}{6}$

solutions of the quadratic equation, $3m^2 + 5m + 1 = 0$? Explain your reasoning.

32. Two stacking bowls are in the shape of hemispheres. They have radii that can be represented by $\sqrt[3]{\frac{3V}{2\pi}}$ and $\sqrt[3]{\frac{V-1}{4\pi}}$, where V represents the volume of the bowl.

- What is the ratio of the larger radius to the smaller radius in simplest form?
- For which volumes is the ratio a real number?

33. MINI LAB

Step 1 Copy and complete the table of values for each equation using technology.

$y = \sqrt{x}$		$y = x^2$	
x	y	x	y
	0	0	
	1	1	
	2	2	
	3	3	
	4	4	

Step 2 Describe any similarities and differences in the patterns of numbers. Compare your answers with those of a classmate.

Step 3 Plot the points for both functions. Compare the shapes of the two graphs. How are the restrictions on the variable for the radical function related to the quadratic function?

Project Corner

Space Exploration

- Earth has a diameter of about 12 800 km and a mass of about 6.0×10^{24} kg. It is about 150 000 000 km from the Sun.
- Artificial gravity is the emulation in outer space of the effects of gravity felt on a planetary surface.
- When travelling into space, it is necessary to overcome the force of gravity. A spacecraft leaving Earth must reach a gravitational escape velocity greater than 11.2 km/s. Research the formula for calculating the escape velocity. Use the formula to determine the escape velocities for the Moon and the Sun.

Radical Equations

Focus on...

- solving equations involving square roots
- determining the roots of a radical equation algebraically
- identifying restrictions on the values for the variable in a radical equation
- modelling and solving problems with radical equations

How do the length and angle of elevation of a ramp affect a skateboarder? How do these measurements affect the height at the top of a ramp? The relationships between measurements are carefully considered when determining safety standards for skate parks, playground equipment, and indoor rock-climbing walls. Architects and engineers also analyse the mathematics involved in these relationships when designing factories and structures such as bridges.

Did You Know?

Shaw Millenium Park, in Calgary, Alberta, is the largest skate park in North America. It occupies about 6000 m^2 , which is about the same area as a CFL football field.



Investigate Radical Equations

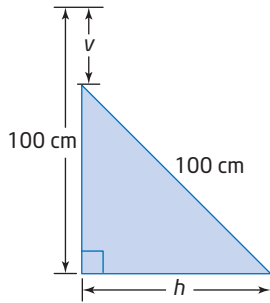
Materials

- three metre sticks
- grid paper

1. To measure vertical distances, place a metre stick vertically against a wall. You may need to tape it in place. To measure horizontal distances, place another metre stick on the ground at the base of the first metre stick, pointing out from the wall.
2. Lean a metre stick against the vertical metre stick. Slide the top of the diagonal metre stick down the wall as the base of it moves away from the wall. Move the base a horizontal distance, h , of 10 cm away from the wall. Then, measure the vertical distance, v , that the top of the metre stick has slid down the wall.
3. Create a table and record values of v for 10-cm increments of h , up to 100 cm.

Horizontal Distance From Wall, h (cm)	Vertical Distance Down Wall, v (cm)
0	
10	
20	

- Analyse the data in the table to determine whether the relationship between v and h is linear or non-linear. Explain how you determined your answer.
- Refer to the diagram. If the diagonal metre stick moves v centimetres down and h centimetres away from the wall, determine the dimensions of the right triangle.



- Write an equation describing v as a function of h . Use your equation to verify two measurements from your table in step 3.



Reflect and Respond

- Estimate the value of h , to the nearest centimetre, when $v = 25$ cm. Verify your estimate using a metre stick.
- As the base of the metre stick passes through the horizontal interval from $(5\sqrt{199} - 5)$ cm to $(5\sqrt{199} + 5)$ cm, what is the vertical change?
- How is solving a **radical equation** similar to solving a linear equation and a quadratic equation? Compare your answers with those of a classmate.

radical equation

- an equation with radicals that have variables in the radicands

Link the Ideas

When solving a radical equation, remember to:

- identify any restrictions on the variable
- identify whether any roots are extraneous by determining whether the values satisfy the original equation

Did You Know?

When multiplying or dividing both sides of an inequality by a negative number, you need to reverse the direction of the inequality symbol.

For example,

$$\begin{aligned}3 - 5n &\geq 0 \\ -5n &\geq -3 \\ \frac{-5n}{-5} &\leq \frac{-3}{-5} \\ n &\leq \frac{3}{5}\end{aligned}$$

Check the solution by isolating the variable in a different way or by substituting values for the variable.

Example 1

Solve an Equation With One Radical Term

- a) State the restrictions on x in $5 + \sqrt{2x - 1} = 12$ if the radical is a real number.
- b) Solve $5 + \sqrt{2x - 1} = 12$.

Solution

- a) For the radical to be a real number, the radicand, $2x - 1$, must be greater than or equal to zero because the index is even. Isolate the variable by performing the same operations on both sides.

$$\begin{aligned}2x - 1 &\geq 0 \\ 2x &\geq 1 \\ x &\geq \frac{1}{2}\end{aligned}$$

For the radical to represent a real number, the variable x must be any real number greater than or equal to $\frac{1}{2}$.

- b) Isolate the radical expression. Square both sides of the equation. Then, solve for the variable.

$$\begin{aligned}5 + \sqrt{2x - 1} &= 12 \\ \sqrt{2x - 1} &= 7 \\ (\sqrt{2x - 1})^2 &= (7)^2 \\ 2x - 1 &= 49 \\ 2x &= 50 \\ x &= 25\end{aligned}$$

What does squaring both sides do?

The value of x meets the restriction in part a).

Check that $x = 25$ is a solution to the original equation.

Left Side	Right Side
$5 + \sqrt{2x - 1}$	12
$= 5 + \sqrt{2(25) - 1}$	
$= 5 + \sqrt{50 - 1}$	
$= 5 + \sqrt{49}$	
$= 5 + 7$	
$= 12$	

$$\text{Left Side} = \text{Right Side}$$

Therefore, the solution is $x = 25$.

Your Turn

Identify any restrictions on y in $-8 + \sqrt{\frac{3y}{5}} = -2$ if the radical is a real number. Then, solve the equation.

Example 2

Radical Equation With an Extraneous Root

What are the restrictions on n if the equation $n - \sqrt{5 - n} = -7$ involves real numbers? Solve the equation.

Solution

$$\begin{aligned}5 - n &\geq 0 \\ 5 &\geq n\end{aligned}$$

Why must the radicand be non-negative?

The value of n can be any real number less than or equal to five.

$$\begin{aligned}n - \sqrt{5 - n} &= -7 && \text{Why, in this case, is the radical isolated} \\ n + 7 &= \sqrt{5 - n} && \text{on the right side of the equal sign?} \\ (n + 7)^2 &= (\sqrt{5 - n})^2 \\ n^2 + 14n + 49 &= 5 - n \\ n^2 + 15n + 44 &= 0\end{aligned}$$

Select a strategy to solve the quadratic equation.

Method 1: Factor the Quadratic Equation

$$\begin{aligned}n^2 + 15n + 44 &= 0 \\ (n + 11)(n + 4) &= 0 && \text{How can you use the zero product property?} \\ n + 11 = 0 &\quad \text{or} \quad n + 4 = 0 \\ n = -11 &\quad \quad n = -4\end{aligned}$$

Method 2: Use the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\begin{aligned}n &= \frac{-15 \pm \sqrt{15^2 - 4(1)(44)}}{2(1)} && \text{How can you identify the values for } a, b, \text{ and } c? \\ n &= \frac{-15 \pm \sqrt{225 - 176}}{2} \\ n &= \frac{-15 + 7}{2} \quad \text{or} \quad n = \frac{-15 - 7}{2} \\ n = -4 &\quad \quad n = -11\end{aligned}$$

Check $n = -4$ and $n = -11$ in the original equation, $n - \sqrt{5 - n} = -7$.

For $n = -4$:

$$\begin{array}{ll}\text{Left Side} & \text{Right Side} \\ n - \sqrt{5 - n} & -7 \\ = -4 - \sqrt{5 - (-4)} & \\ = -4 - 3 & \\ = -7 & \end{array}$$

Left Side = Right Side

For $n = -11$:

$$\begin{array}{ll}\text{Left Side} & \text{Right Side} \\ n - \sqrt{5 - n} & -7 \\ = -11 - \sqrt{5 - (-11)} & \\ = -11 - 4 & \\ = -15 & \end{array}$$

Left Side \neq Right Side

The solution is $n = -4$. The value $n = -11$ is extraneous. Extraneous roots occur because squaring both sides and solving the quadratic equation may result in roots that do not satisfy the original equation.

Your Turn

State the restrictions on the variable in $m - \sqrt{2m + 3} = 6$ if the equation involves real numbers. Then, solve the equation.

Example 3

Solve an Equation With Two Radicals

Solve $7 + \sqrt{3x} = \sqrt{5x + 4} + 5$, $x \geq 0$. Check your solution.

Solution

Isolate one radical and then square both sides.

$$\begin{aligned} 7 + \sqrt{3x} &= \sqrt{5x + 4} + 5 \\ 2 + \sqrt{3x} &= \sqrt{5x + 4} && \text{Why is it beneficial to isolate} \\ (2 + \sqrt{3x})^2 &= (\sqrt{5x + 4})^2 && \text{the more complex radical first?} \end{aligned}$$

$$4 + 4\sqrt{3x} + 3x = 5x + 4$$

Isolate the remaining radical, square both sides, and solve.

$$\begin{aligned} 4\sqrt{3x} &= 2x \\ (4\sqrt{3x})^2 &= (2x)^2 \\ 16(3x) &= 4x^2 \\ 48x &= 4x^2 \\ 0 &= 4x^2 - 48x \\ 0 &= 4x(x - 12) \\ 4x &= 0 \quad \text{or} \quad x - 12 = 0 \\ x &= 0 \quad \quad \quad x = 12 \end{aligned}$$

Check $x = 0$ and $x = 12$ in the original equation.

For $x = 0$:

Left Side	Right Side
$7 + \sqrt{3x}$	$\sqrt{5x + 4} + 5$
$= 7 + \sqrt{3(0)}$	$= \sqrt{5(0) + 4} + 5$
$= 7 + \sqrt{0}$	$= 2 + 5$
$= 7$	$= 7$

Left Side = Right Side

For $x = 12$:

Left Side	Right Side
$7 + \sqrt{3x}$	$\sqrt{5x + 4} + 5$
$= 7 + \sqrt{3(12)}$	$= \sqrt{5(12) + 4} + 5$
$= 7 + \sqrt{36}$	$= \sqrt{64} + 5$
$= 13$	$= 13$

Left Side = Right Side

The solutions are $x = 0$ and $x = 12$.

Your Turn

Solve $\sqrt{3 + j} + \sqrt{2j - 1} = 5$, $j \geq \frac{1}{2}$.

Example 4

Solve Problems Involving Radical Equations

What is the speed, in metres per second, of a 0.4-kg football that has 28.8 J of kinetic energy? Use the kinetic energy formula, $E_k = \frac{1}{2}mv^2$, where E_k represents the kinetic energy, in joules; m represents mass, in kilograms; and v represents speed, in metres per second.

Solution

Method 1: Rearrange the Equation

$$E_k = \frac{1}{2}mv^2$$

$$\frac{2E_k}{m} = v^2$$

$$\pm\sqrt{\frac{2E_k}{m}} = v$$

Why is the symbol \pm included in this step?

Substitute $m = 0.4$ and $E_k = 28.8$ into

the radical equation, $v = \sqrt{\frac{2E_k}{m}}$.

$$v = \sqrt{\frac{2(28.8)}{0.4}}$$

Why is only the positive root considered?

$$v = 12$$

The speed of the football is 12 m/s.

Method 2: Substitute the Given Values and Evaluate

$$E_k = \frac{1}{2}mv^2$$

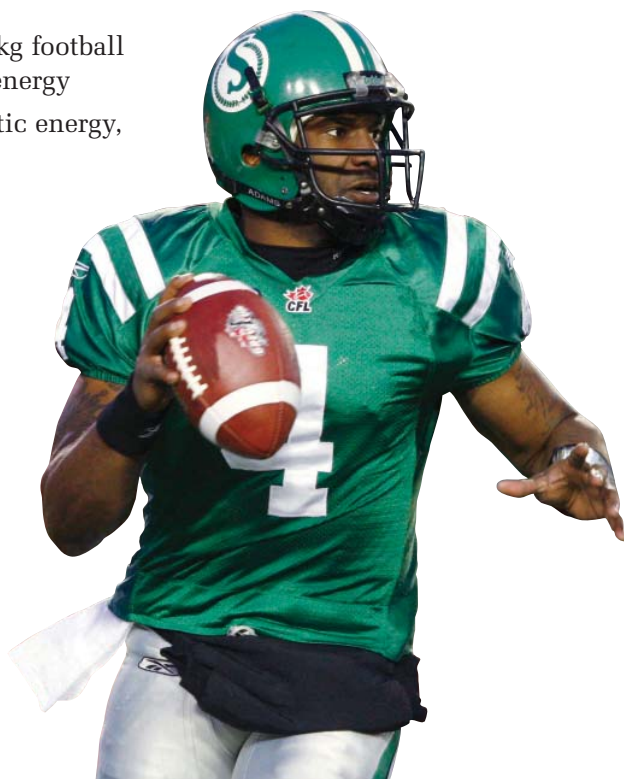
$$28.8 = \frac{1}{2}(0.4)v^2$$

$$144 = v^2$$

$$\pm 12 = v$$

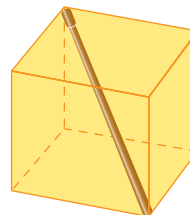
The speed of the football is 12 m/s.

Why is only the positive root considered?



Your Turn

Josh is shipping several small musical instruments in a cube-shaped box, including a drumstick which just fits diagonally in the box. Determine the formula for the length, d , in centimetres, of the drumstick in terms of the area, A , in square centimetres, of one face of the box. What is the area of one face of a cube-shaped box that holds a drumstick of length 23.3 cm? Express your answer to the nearest square centimetre.



Key Ideas

- You can model some real-world relationships with radical equations.
- When solving radical equations, begin by isolating one of the radical terms.
- To eliminate a square root, raise both sides of the equation to the exponent two. For example, in $3 = \sqrt{c + 5}$, square both sides.

$$3^2 = (\sqrt{c + 5})^2$$

$$9 = c + 5$$

$$4 = c$$

- To identify whether a root is extraneous, substitute the value into the original equation. Raising both sides of an equation to an even exponent may introduce an extraneous root.
- When determining restrictions on the values for variables, consider the following:
 - Denominators cannot be equal to zero.
 - For radicals to be real numbers, radicands must be non-negative if the index is an even number.

Check Your Understanding

Practise

Determine any restrictions on the values for the variable in each radical equation, unless given.

1. Square each expression.

a) $\sqrt{3z}$, $z \geq 0$

b) $\sqrt{x - 4}$, $x \geq 4$

c) $2\sqrt{x + 7}$, $x \geq -7$

d) $-4\sqrt{9 - 2y}$, $\frac{9}{2} \geq y$

2. Describe the steps to solve the equation $\sqrt{x} + 5 = 11$, where $x \geq 0$.

3. Solve each radical equation. Verify your solutions and identify any extraneous roots.

a) $\sqrt{2x} = 3$

b) $\sqrt{-8x} = 4$

c) $7 = \sqrt{5 - 2x}$

4. Solve each radical equation. Verify your solutions.

a) $\sqrt{z} + 8 = 13$

b) $2 - \sqrt{y} = -4$

c) $\sqrt{3x} - 8 = -6$

d) $-5 = 2 - \sqrt{-6m}$

5. In the solution to $k + 4 = \sqrt{-2k}$, identify whether either of the values, $k = -8$ or $k = -2$, is extraneous. Explain your reasoning.

6. Isolate each radical term. Then, solve the equation.

a) $-3\sqrt{n - 1} + 7 = -14$, $n \geq 1$

b) $-7 - 4\sqrt{2x - 1} = 17$, $x \geq \frac{1}{2}$

c) $12 = -3 + 5\sqrt{8 - x}$, $x \leq 8$

7. Solve each radical equation.

- a) $\sqrt{m^2 - 3} = 5$
- b) $\sqrt{x^2 + 12x} = 8$
- c) $\sqrt{\frac{q^2}{2}} + 11 = q - 1$
- d) $2n + 2\sqrt{n^2 - 7} = 14$

8. Solve each radical equation.

- a) $5 + \sqrt{3x - 5} = x$
- b) $\sqrt{x^2 + 30x} = 8$
- c) $\sqrt{d + 5} = d - 1$
- d) $\sqrt{\frac{j + 1}{3}} + 5j = 3j - 1$

9. Solve each radical equation.

- a) $\sqrt{2k} = \sqrt{8}$
- b) $\sqrt{-3m} = \sqrt{-7m}$
- c) $5\sqrt{\frac{j}{2}} = \sqrt{200}$
- d) $5 + \sqrt{n} = \sqrt{3n}$

10. Solve.

- a) $\sqrt{z + 5} = \sqrt{2z - 1}$
- b) $\sqrt{6y - 1} = \sqrt{-17 + y^2}$
- c) $\sqrt{5r - 9} - 3 = \sqrt{r + 4} - 2$
- d) $\sqrt{x + 19} + \sqrt{x - 2} = 7$

Apply

11. By inspection, determine which one of the following equations will have an extraneous root. Explain your reasoning.

$$\begin{aligned}\sqrt{3y - 1} - 2 &= 5 \\ 4 - \sqrt{m + 6} &= -9 \\ \sqrt{x + 8} + 9 &= 2\end{aligned}$$

12. The following steps show how Jerry solved the equation $3 + \sqrt{x + 17} = x$. Is his work correct? Explain your reasoning and provide a correct solution if necessary.

Jerry's Solution

$$\begin{aligned}3 + \sqrt{x + 17} &= x \\ \sqrt{x + 17} &= x - 3 \\ (\sqrt{x + 17})^2 &= (x - 3)^2 \\ x + 17 &= x^2 - 9 \\ 0 &= x^2 - x - 26 \\ x &= \frac{1 \pm \sqrt{1 + 104}}{2} \\ x &= \frac{1 \pm \sqrt{105}}{2}\end{aligned}$$

13. Collision investigators can approximate the initial velocity, v , in kilometres per hour, of a car based on the length, l , in metres, of the skid mark. The formula $v = 12.6\sqrt{l} + 8$, $l \geq 0$, models the relationship. What length of skid is expected if a car is travelling 50 km/h when the brakes are applied? Express your answer to the nearest tenth of a metre.



14. In 1805, Rear-Admiral Beaufort created a numerical scale to help sailors quickly assess the strength of the wind. The integer scale ranges from 0 to 12. The wind scale, B , is related to the wind velocity, v , in kilometres per hour, by the formula $B = 1.33\sqrt{v + 10.0} - 3.49$, $v \geq -10$.

- a) Determine the wind scale for a wind velocity of 40 km/h.
- b) What wind velocity results in a wind scale of 3?



Web Link

To learn more about the Beaufort scale, go to www.mhrprecalc11.ca and follow the links.

15. The mass, m , in kilograms, that a beam with a fixed width and length can support is related to its thickness, t , in centimetres. The formula is

$$t = \frac{1}{5}\sqrt{\frac{m}{3}}, m \geq 0.$$

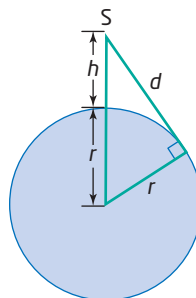
If a beam is 4 cm thick, what mass can it support?



16. Two more than the square root of a number, n , is equal to the number. Model this situation using a radical equation. Determine the value(s) of n algebraically.
17. The speed, v , in metres per second, of water being pumped into the air to fight a fire is the square root of twice the product of the maximum height, h , in metres, and the acceleration due to gravity. At sea level, the acceleration due to gravity is 9.8 m/s^2 .
- Write the formula that models the relationship between the speed and the height of the water.
 - Suppose the speed of the water being pumped is 30 m/s. What expected height will the spray reach?
 - A local fire department needs to buy a pump that reaches a height of 60 m. An advertisement for a pump claims that it can project water at a speed of 35 m/s. Will this pump meet the department's requirements? Justify your answer.



18. The distance d , in kilometres, to the horizon from a height, h , in kilometres, can be modelled by the formula $d = \sqrt{2rh + h^2}$, where r represents Earth's radius, in kilometres. A spacecraft is 200 km above Earth, at point S. If the distance to the horizon from the spacecraft is 1609 km, what is the radius of Earth?



Extend

19. Solve for a in the equation $\sqrt{3x} = \sqrt{ax} + 2$, $a \geq 0$, $x > 0$.
20. Create a radical equation that results in the following types of solution. Explain how you arrived at your equation.
- one extraneous solution and no valid solutions
 - one extraneous solution and one valid solution
21. The time, t , in seconds, for an object to fall to the ground is related to its height, h , in metres, above the ground. The formulas for determining this time are $t_m = \sqrt{\frac{h}{1.8}}$ for the moon and $t_E = \sqrt{\frac{h}{4.9}}$ for Earth. The same object is dropped from the same height on both the moon and Earth. If the difference in times for the object to reach the ground is 0.5 s, determine the height from which the object was dropped. Express your answer to the nearest tenth of a metre.
22. Refer to #18. Use the formula $d = \sqrt{2rh + h^2}$ to determine the height of a spacecraft above the moon, where the radius is 1740 km and the distance to the horizon is 610 km. Express your answer to the nearest kilometre.

- 23.** The profit, P , in dollars, of a business can be expressed as $P = -n^2 + 200n$, where n represents the number of employees.
- What is the maximum profit? How many employees are required for this value?
 - Rewrite the equation by isolating n .
 - What are the restrictions on the radical portion of your answer to part b)?
 - What are the domain and range for the original function? How does your answer relate to part c)?

Create Connections

- 24.** Describe the similarities and differences between solving a quadratic equation and solving a radical equation.
- 25.** Why are extraneous roots sometimes produced when solving radical equations? Include an example and show how the root was produced.
- 26.** An equation to determine the annual growth rate, r , of a population of moose in Wells Gray Provincial Park, British Columbia, over a 3-year period is
- $$r = -1 + \sqrt[3]{\frac{P_f}{P_i}}, P_f \geq 0, P_i > 0.$$
- In the equation, P_i represents the initial population 5 years ago and P_f represents the final population after 3 years.
- If $P_i = 320$ and $P_f = 390$, what is the annual growth rate? Express your answer as a percent to the nearest tenth.
 - Rewrite the equation by isolating P_f .
 - Determine the four populations of moose over this 3-year period.
 - What kind of sequence does the set of populations in part c) represent?



- 27. MINI LAB** A continued radical is a series of nested radicals that may be infinite but has a finite rational result. Consider the following continued radical:

$$\sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \dots}}}}$$

- Step 1** Using a calculator or spreadsheet software, determine a decimal approximation for the expressions in the table.

Number of Nested Radicals	Expression	Decimal Approximation
1	$\sqrt{6 + \sqrt{6}}$	
2	$\sqrt{6 + \sqrt{6 + \sqrt{6}}}$	
3	$\sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}}$	
4		
5		
6		
7		
8		
9		

- Step 2** From your table, predict the value of the expression
- $$\sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \dots}}}}$$
- Step 3** Let $x = \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \dots}}}}$. Solve the equation algebraically.
- Step 4** Check your result with a classmate. Why does one of the roots need to be rejected?
- Step 5** Generate another continued radical expression that will result in a finite real-number solution. Does your answer have a rational or an irrational root?
- Step 6** Exchange your radical expression in step 5 with a classmate and solve their problem.

Chapter 5 Review

5.1 Working With Radicals, pages 272-281

- Convert each mixed radical to an entire radical.
 - $8\sqrt{5}$
 - $-2\sqrt[5]{3}$
 - $3y^3\sqrt{7}$
 - $-3z(\sqrt[3]{4z})$
- Convert each entire radical to a mixed radical in simplest form.
 - $\sqrt{72}$
 - $3\sqrt{40}$
 - $\sqrt{27m^2}$, $m \geq 0$
 - $\sqrt[3]{80x^5y^6}$
- Simplify.
 - $-\sqrt{13} + 2\sqrt{13}$
 - $4\sqrt{7} - 2\sqrt{112}$
 - $-\sqrt[3]{3} + \sqrt[3]{24}$
- Simplify radicals and collect like terms. State any restrictions on the values for the variables.
 - $4\sqrt{45x^3} - \sqrt{27x} + 17\sqrt{3x} - 9\sqrt{125x^3}$
 - $\frac{2}{5}\sqrt{44a} + \sqrt{144a^3} - \frac{\sqrt{11a}}{2}$
- Which of the following expressions is not equivalent to $8\sqrt{7}$?
 $2\sqrt{112}$, $\sqrt{448}$, $3\sqrt{42}$, $4\sqrt{28}$
 Explain how you know without using technology.
- Order the following numbers from least to greatest: $3\sqrt{7}$, $\sqrt{65}$, $2\sqrt{17}$, 8
- The speed, v , in kilometres per hour, of a car before a collision can be approximated from the length, d , in metres, of the skid mark left by the tire. On a dry day, one formula that approximates this speed is $v = \sqrt{169d}$, $d \geq 0$.
 - Rewrite the formula as a mixed radical.
 - What is the approximate speed of a car if the skid mark measures 13.4 m? Express your answer to the nearest kilometre per hour.

- The city of Yorkton, Saskatchewan, has an area of 24.0 km^2 . If this city were a perfect square, what would its exact perimeter be? Express your answer as a mixed radical in simplest form.



- State whether each equation is true or false. Justify your reasoning.
 - $-3^2 = \pm 9$
 - $(-3)^2 = 9$
 - $\sqrt{9} = \pm 3$

5.2 Multiplying and Dividing Radical Expressions, pages 282-293

- Multiply. Express each product as a radical in simplest form.
 - $\sqrt{2}(\sqrt{6})$
 - $(-3f\sqrt{15})(2f^3\sqrt{5})$
 - $(\sqrt[4]{8})(3\sqrt[4]{18})$
- Multiply and simplify. Identify any restrictions on the values for the variable in part c).
 - $(2 - \sqrt{5})(2 + \sqrt{5})$
 - $(5\sqrt{3} - \sqrt{8})^2$
 - $(a + 3\sqrt{a})(a + 7\sqrt{4a})$
- Are $x = \frac{5 + \sqrt{17}}{2}$ and $x = \frac{5 - \sqrt{17}}{2}$ a conjugate pair? Justify your answer. Are they solutions of the quadratic equation $x^2 - 5x + 2 = 0$? Explain.

13. Rationalize each denominator.

a) $\frac{\sqrt{6}}{\sqrt{12}}$

b) $\frac{-1}{\sqrt[3]{25}}$

c) $-4\sqrt{\frac{2a^2}{9}}, a \geq 0$

14. Rationalize each denominator. State any restrictions on the values for the variables.

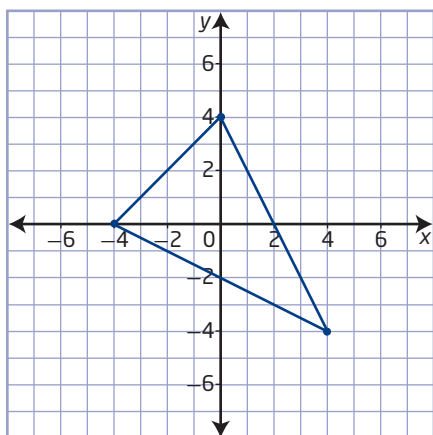
a) $\frac{-2}{4 - \sqrt{3}}$

b) $\frac{\sqrt{7}}{2\sqrt{5} - \sqrt{7}}$

c) $\frac{18}{6 + \sqrt{27m}}$

d) $\frac{a + \sqrt{b}}{a - \sqrt{b}}$

15. What is the exact perimeter of the triangle?



16. Simplify.

a) $\left(\frac{-5\sqrt{3}}{\sqrt{6}}\right)\left(\frac{-\sqrt{7}}{3\sqrt{21}}\right)$

b) $\left(\frac{2a\sqrt{a^3}}{9}\right)\left(\frac{12}{-\sqrt{8a}}\right)$

17. The area of a rectangle is 12 square units. The width is $(4 - \sqrt{2})$ units. Determine an expression for the length of the rectangle in simplest radical form.

5.3 Radical Equations, pages 294–303

18. Identify the values of x for which the radicals are defined. Solve for x and verify your answers.

a) $-\sqrt{x} = -7$

b) $\sqrt{4 - x} = -2$

c) $5 - \sqrt{2x} = -1$

d) $1 + \sqrt{\frac{7x}{3}} = 8$

19. Solve each radical equation. Determine any restrictions on the values for the variables.

a) $\sqrt{5x - 3} = \sqrt{7x - 12}$

b) $\sqrt{y - 3} = y - 3$

c) $\sqrt{7n + 25} - n = 1$

d) $\sqrt{8 - \frac{m}{3}} = \sqrt{3m} - 4$

e) $\sqrt[3]{3x - 1} + 7 = 3$

20. Describe the steps in your solution to #19c). Explain why one of the roots was extraneous.

21. On a calm day, the distance, d , in kilometres, that the coast guard crew on the Coast Guard cutter Vakta can see to the horizon depends on their height, h , in metres, above the water. The formula $d = \sqrt{\frac{3h}{2}}, h \geq 0$ models this relationship. What is the height of the crew above the water if the distance to the horizon is 7.1 km?



Did You Know?

The Vakta is a 16.8-m cutter used by Search and Rescue to assist in water emergencies on Lake Winnipeg. It is based in Gimli, Manitoba.

Chapter 5 Practice Test

Multiple Choice

For #1 to #6, choose the best answer.

1. What is the entire radical form of $-3(\sqrt[3]{2})$?

A $\sqrt[3]{54}$ B $\sqrt[3]{-54}$
C $\sqrt[3]{-18}$ D $\sqrt[3]{18}$

2. What is the condition on the variable in $2\sqrt{-7n}$ for the radicand to be a real number?

A $n \geq 7$ B $n \leq -7$
C $n \geq 0$ D $n \leq 0$

3. What is the simplest form of the sum $-2x\sqrt{6x} + 5x\sqrt{6x}$, $x \geq 0$?

A $3\sqrt{6x}$ B $6\sqrt{12x}$
C $3x\sqrt{6x}$ D $6x\sqrt{12}$

4. What is the product of $\sqrt{540}$ and $\sqrt{6y}$, $y \geq 0$, in simplest form?

A $3y\sqrt{360}$ B $6y\sqrt{90}$
C $10\sqrt{32y}$ D $18\sqrt{10y}$

5. Determine any root of the equation, $x + 7 = \sqrt{23 - x}$, where $x \leq 23$.

A $x = 13$
B $x = -2$
C $x = 2$ and 13
D $x = -2$ and -13

6. Suppose $\frac{5}{7}\sqrt{\frac{3}{2}}$ is written in simplest form as $a\sqrt{b}$, where a is a real number and b is an integer. What is the value of b ?

A 2 B 3
C 6 D 14

Short Answer

7. Order the following numbers from least to greatest:

$3\sqrt{11}$, $5\sqrt{6}$, $9\sqrt{2}$, $\sqrt{160}$

8. Express as a radical in simplest form.

$$\frac{(2\sqrt{5n})(3\sqrt{8n})}{1 - 12\sqrt{2}}, n \geq 0.$$

9. Solve $3 - x = \sqrt{x^2 - 5}$. State any extraneous roots that you found. Identify the values of x for which the radical is defined.

10. Solve $\sqrt{9y + 1} = 3 + \sqrt{4y - 2}$, $y \geq \frac{1}{2}$. Verify your solution. Justify your method. Identify any extraneous roots that you found.

11. Masoud started to simplify $\sqrt{450}$ by rewriting 450 as a product of prime factors: $\sqrt{2(3)(3)(5)(5)}$. Explain how he can convert his expression to a mixed radical.

12. For sailboats to travel into the wind, it is sometimes necessary to tack, or move in a zigzag pattern. A sailboat in Lake Winnipeg travels 4 km due north and then 4 km due west. From there the boat travels 5 km due north and then 5 km due west. How far is the boat from its starting point? Express your answer as a mixed radical.



13. You wish to rationalize the denominator in each expression. By what number will you multiply each expression? Justify your answer.

a) $\frac{4}{\sqrt{6}}$
b) $\frac{22}{\sqrt{y-3}}$
c) $\frac{2}{\sqrt[3]{7}}$

14. For diamonds of comparable quality, the cost, C , in dollars, is related to the mass, m , in carats, by the formula

$m = \sqrt{\frac{C}{700}}$, $C \geq 0$. What is the cost of a 3-carat diamond?

Did You Know?

Snap Lake Mine is 220 km northeast of Yellowknife, Northwest Territories. It is the first fully underground diamond mine in Canada.



15. Teya tries to rationalize the denominator in the expression $\frac{5\sqrt{2} + \sqrt{3}}{4\sqrt{2} - \sqrt{3}}$. Is Teya correct? If not, identify and explain any errors she made.

Teya's Solution

$$\begin{aligned} \frac{5\sqrt{2} + \sqrt{3}}{4\sqrt{2} - \sqrt{3}} &= \left(\frac{5\sqrt{2} + \sqrt{3}}{4\sqrt{2} - \sqrt{3}} \right) \left(\frac{4\sqrt{2} + \sqrt{3}}{4\sqrt{2} + \sqrt{3}} \right) \\ &= \frac{20\sqrt{4} + 5\sqrt{6} + 4\sqrt{6} + \sqrt{9}}{32 - 3} \\ &= \frac{40 + 9\sqrt{6} + 3}{32 - 3} \\ &= \frac{43 + 9\sqrt{6}}{29} \end{aligned}$$

16. A right triangle has one leg that measures 1 unit.
- Model the length of the hypotenuse using a radical equation.
 - The length of the hypotenuse is 11 units. What is the length of the unknown leg? Express your answer as a mixed radical in simplest form.

Extended Response

17. A 100-W light bulb operates with a current of 0.5 A. The formula relating current, I , in amperes (A); power, P , in watts (W); and resistance, R , in ohms (Ω), is $I = \sqrt{\frac{P}{R}}$.
- Isolate R in the formula.
 - What is the resistance in the light bulb?
18. Sylvie built a model of a cube-shaped house.
- Express the edge length of a cube in terms of the surface area using a radical equation.
 - Suppose the surface area of the cube in Sylvie's model is 33 cm^2 . Determine the exact edge length in simplest form.
 - If the surface area of a cube doubles, by what scale factor will the edge length change?
19. Beverley invested \$3500 two years ago. The investment earned compound interest annually according to the formula $A = P(1 + i)^n$. In the formula, A represents the final amount of the investment, P represents the principal or initial amount, i represents the interest rate per compounding period, and n represents the number of compounding periods. The current amount of her investment is \$3713.15.
- Model Beverley's investment using the formula.
 - What is the interest rate? Express your answer as a percent.